

BONNYCASTLE'S

MENSURATION.

FOR THE YEAR 1877

AND THE YEAR 1878

AN
INTRODUCTION
TO
MENSURATION,
AND
PRACTICAL GEOMETRY.

WITH NOTES, CONTAINING THE REASON OF EVERY
RULE, CONCISELY AND CLEARLY DEMONSTRATED.

By JOHN BONNYCASTLE,
Author of the SCHOLAR'S GUIDE TO ARITHMETIC.



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MUSEUM

AND

PRACTICAL GEOMETRY

WITH NOTES CONTAINING THE HISTORY OF THE
ART, AND A COLLECTION OF PROBLEMS

BY JOHN ROBERTSON

OF THE UNIVERSITY OF GLASGOW

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P R E F A C E.

THE ART OF MEASURING, like all other useful inventions, was the offspring of want and necessity; and had its origin in those remote ages of antiquity, which are far beyond the reach of credible and authentic history.

Egypt, the fruitful mother of almost all the liberal sciences, is imagined likewise to have given birth to GEOMETRY OF MENSURATION; and it is to the inundations of the Nile that we are said to be indebted for this most perfect and delightful branch of human knowledge.

After the overflowings of the river had deluged the whole country, and all artificial boundaries and landmarks were destroyed, there could have been no other method of ascertaining individual property, than by a previous knowledge of its figure and dimensions. From this circumstance, therefore, it appears highly probable that Geometry was first known and cultivated by the ancient Egyptians; as being the only science that could administer to their wants, and furnish them with the assistance they required. The name itself signifies properly nothing more than the *art of measuring the earth*; which serves still further to confirm this opinion; especially as it is well known that most of the ancient mathematicians applied their geometrical knowledge entirely to that purpose; and that even the Elements of Euclid, as they now stand, are only the theory from whence we obtain the rules and precepts of our present more mechanical practice.

But to trace the sciences to their first rude beginnings, is a matter only of learned curiosity. It is of much more consequence to the rising generation to be informed that, in their present improved state, they are exceedingly useful and important. And in this respect, the art I have undertaken to elucidate is inferior to none, arithmetic only excepted. Its use in most of the different branches of the Mathematics is so general and extensive, that it may be justly considered as the mother and mistress of them all, and the source from whence were derived the various properties and principles to which they owe their existence.

As a testimony of this superior excellence, I need only mention a few of those who have studied and improved it. And in this illustrious catalogue we have the names of Euclid, Archimedes, Thales, Anaxagoras, Pythagoras, Plato, Apollonius, Philo and Ptolemy, amongst the ancients; and Huygens, Wallis, Gregory, Halley, Bernoulli, and the immortal Newton, amongst the moderns; all of whom applied themselves, with great diligence, to particular parts of it; and the latter especially have carried them to the greatest degree of perfection.

Having thus shewn the degree of estimation in which this art was held by some of the first characters that ever the world produced, any further encomium on its merits will, in general, I apprehend, be thought unnecessary. But, for the sake of young people, and those of a confined education, it may not be amiss to give a few more instances of its advantage, and shew that its importance in trade and business is not inferior to its dignity as a science. Artificers of almost all denominations are indebted to this invention for the establishment of their several occupations, and the perfection and value of their workmanship. Without its assistance all the great and noble works of Art would have been imperfect and
useless.

useless. By this means the architect lays down his plan, and erects his edifice. Bridges are built over large rivers. Ships are constructed. And property of all kinds is accurately measured, and justly estimated. In short, most of the elegancies and conveniencies of life owe their existence to this art, and will be multiplied in proportion as it is well understood, and properly practised.

From this view of the subject, it is hardly to be accounted for, that, in a commercial nation, like our own, an art of such general application should have been so greatly neglected. Mechanics of all kinds, it is well known, are but ill acquainted with its principles; and those who have been the best qualified to afford them any assistance, have thought it beneath their attention. Till within a few years past there could not be found a regular treatise upon this subject in the English language. Some particular branches, it is true, had been greatly cultivated and improved. But these were only to be found in their miscellaneous state, interspersed through a number of large volumes, in the possession of but few, and in a form and language totally unintelligible to those for whom they were more immediately necessary.

Dr. Hutton was the first person, in this country, who undertook to collect these scattered fragments, and to treat of the subject in a scientific, methodical manner. Before this time, Mr. Robertson's was the only book, of any value, that could be consulted, either by the artizan or mathematician; and had he given the theory as well as the practice, and divested his rules and examples of their algebraical form, there would have been no want of any other elementary treatise.

To these two writers I am greatly indebted for many things in the following pages, and am ready

to

to acknowledge, that I have used an unreserved freedom in selecting from their works, wherever I found them to answer my purpose. To Dr. Hutton I am particularly obliged, and am so far from desiring to supersede the use of his performance by this publication, that I only wish it to be thought a useful introduction to it. His treatise is excellent in its kind; and had it been as well calculated for the use of the uninformed Artist as it is for the Mathematician, the following compendium had certainly never been published.

The method I have observed, in composing this work, is that which was used in the "*Scholar's Guide to Arithmetic*;" and, as my object has been to facilitate the acquirement of the same kind of useful knowledge, I am not without hopes of its being received with equal candour and approbation.

In school-books, and those designed for the use of mere novices, it has always appeared to me, that plain and concise rules, with proper exercises, are entirely sufficient for the purpose. In science, as well as in morals, example will ever enforce and illustrate precept; and for this reason an operation, wrought out at full length, will be found of more service to beginners than all the tedious directions and observations that can possibly be given them. From constant experience I have been confirmed in this idea; and it is in pursuance of it that I have formed the plan of this publication. I have not been ambitious of adding much new matter to the subject; but only to arrange and methodize it in a manner more easy and rational than had been done before.

The text part of the work contains the rules in words at length, with examples to exercise them; and, in order that the learner may not be perplexed and interrupted in his progress, the remarks and demonstrations are confined to the notes, and may be consulted or not, as shall be thought necessary.

To

To those who would wish not to take things upon trust, but to be acquainted with the grounds and *rationale* of the operations they perform, the demonstrations will be found extremely serviceable; and for this purpose I have endeavoured to make them as easy as the nature of the subject would admit. But they can be consulted only by such as have made a previous acquaintance with several other branches of mathematical learning.

Some of the most difficult rules relating to the surfaces of solids, &c. could not be conveniently given, but by means of algebraical theorems; and as this was foreign to my purpose, I have not scrupled to omit them; being well persuaded that what is done upon that head will be fully sufficient to answer most practical purposes.

In the Practical Geometry, prefixed to this treatise, such problems only are introduced as were known to be most intimately connected with the subject. And as this part of the work is a proper and necessary introduction to the rest, I have spared no pains in making it as clear and intelligible as possible.

Upon the whole, I have endeavoured to consult the wants of the learner, more than those of the man of science. And if I have succeeded in this respect, my purpose is answered. I have not sought for reputation as a mathematician, but only to be useful as a tutor.

C O N T E N T S.

The first part of the work is devoted to a general
 introduction of the subject, and to a statement of the
 objects and scope of the inquiry. The second part
 contains a history of the subject, and a statement of
 the progress of the science. The third part is
 devoted to a statement of the principles of the
 subject, and to a statement of the methods of
 investigation. The fourth part is devoted to a
 statement of the results of the inquiry, and to a
 statement of the conclusions to which the inquiry
 has led. The fifth part is devoted to a
 statement of the applications of the subject, and
 to a statement of the value of the subject.

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ERRATA.

Page 40, line 23, *for* 5 fe. 5 in. *read* 5 fe. 9 in.
The answers to all the examples belonging to the
rule in p. 44, are double of what they ought to be.
For BF, in the example page 96, *read* OF. Sup-
ply AFE in the figure page 130.

EXTRACT

The following is a list of the names of the persons who have been appointed to the various offices of the County of New York, for the year 1880. The names are given in alphabetical order, and the offices are given in the order in which they are held. For a full and complete list of the names of the persons who have been appointed to the various offices of the County of New York, for the year 1880, see the full and complete list of the names of the persons who have been appointed to the various offices of the County of New York, for the year 1880, in the full and complete list of the names of the persons who have been appointed to the various offices of the County of New York, for the year 1880.

PRACTICAL GEOMETRY.

DEFINITIONS.

I. **G**EOMETRY is that science which treats of the descriptions and properties of magnitudes in general.

II. A *point* is that which has no parts, or dimensions.

III. A *line* is length without breadth.

IV. The *bounds* or *extremes* of a line are points.

V. A *right line* is that which lies evenly between its extreme points.

VI. A *superficies* is that which has length and breadth only.



A. III.

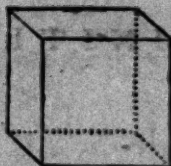
B

VII. The

VII. The *bounds* or *extremes* of a superficies are lines.

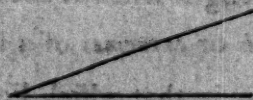
VIII. A *plane superficies* is that which is perfectly flat and even; or which touches, in every part, any right line whatever that can be drawn in that superficies.

IX. A *solid* is that which has length, breadth, and thickness.

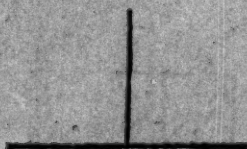


X. The *bounds* of a solid are surfaces.

XI. A *plane rectilinear angle* is the inclination or opening of two right lines meeting in a point.



XI. One line is said to be *perpendicular* to another line, when it makes the angles on both sides of it equal to each other.

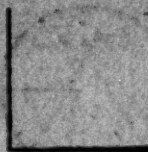


XII. A

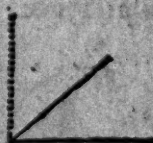
PRACTICAL GEOMETRY.

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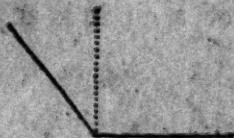
XII. A *right angle* is that which is formed by two lines that are perpendicular to each other.



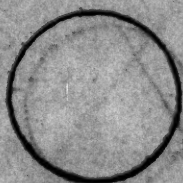
XIII. An *acute angle* is that which is less than a right angle.



XIV. An *obtuse angle* is that which is greater than a right angle.



XV. A *circle* is a plane figure bounded by one uniform curve line, called the *circumference*, which is every where equally distant from a point within the figure called the *center*.

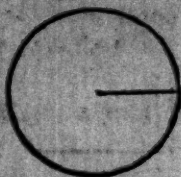


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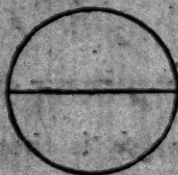
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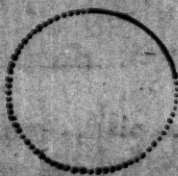
XVI. The *radius* of a circle is a right line drawn from the center to the circumference.



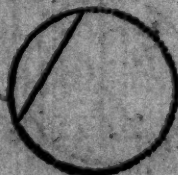
XVII. The *diameter* of a circle is a right line, which passes through the center, and is terminated both ways by the circumference.



XVIII. An *arc* of a circle is any part of the circumference.



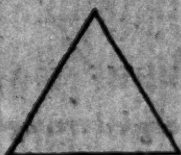
XIX. A *chord* is a right line joining the extremes of an arc.



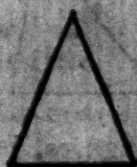
XX. All

XX. All plane figures bounded by three right lines are called *triangles*.

XXI. An *equilateral triangle* is that whose three sides are all equal.



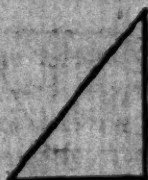
XXII. An *isosceles triangle* is that which has only two sides equal.



XXIII. A *scalene triangle* is that which has all its three sides unequal.



XXIV. A *right angled triangle* is that which has one right angle.

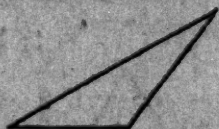


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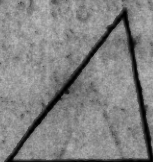
XXV. An

6 PRACTICAL GEOMETRY.

XXV. An *obtuse angled triangle* is that which has one obtuse angle.

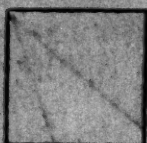


XXVI. An *acute angled triangle* is that which has all its angles acute.

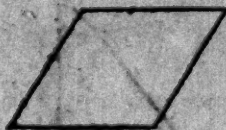


XXVII. All plane figures, bounded by four right lines, are called *quadrangles*, or *quadrilaterals*.

XXVIII. A *square* is a quadrangle, whose sides are all equal, and its angles all right angles.



XXIX. A *rhombus* is a quadrangle whose sides are all equal, but its angles not right angles.

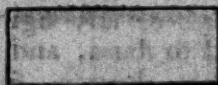


XXX. A

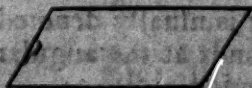
XXX. A *parallelogram* is a quadrangle whose opposite sides are parallel.



XXXI. A *rectangle* is a parallelogram whose angles are all right angles.

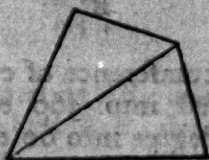


XXXII. A *rhomboid* is a parallelogram whose angles are not right angles.



XXXIII. All other four-sided figures, besides these, are called *trapeziums*.

XXXIV. A right line joining any two opposite angles of a four-sided figure is called the *diagonal*.



XXXV. All plane figures contained under more than four sides are called *polygons*.

XXXVI. Polygons having five sides are called *pentagons*; those having six sides are called *hexagons*; and those of seven sides *heptagons*; and so on.

8 PRACTICAL GEOMETRY.

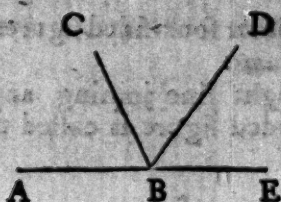
XXXVII. A *regular polygon* is that whose angles, as well as sides, are all equal.

XXXVIII. *Parallel right lines* are such as are every where at an equal distance, and if infinitely produced would never meet.

XXXIX. The *base* of any figure is that side on which it is supposed to stand, and the *altitude* is the perpendicular falling thereon from the opposite angle.

XL. In a right angled triangle the side opposite to the right angle is called the *hypotenuse*, and the other two sides are called *legs*.

XLI. An angle is usually denoted by three letters, and that which stands at the angular point is always to be read in the middle.



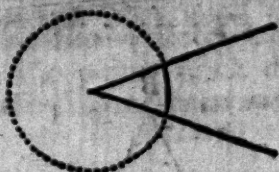
* XLII. The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees*; and each degree into 60 equal parts, called *minutes*, &c.

XLIII. The *measure* of any right lined angle is an arc of a circle contained between the two lines

* This and the following definition are used only in Practical Geometry.

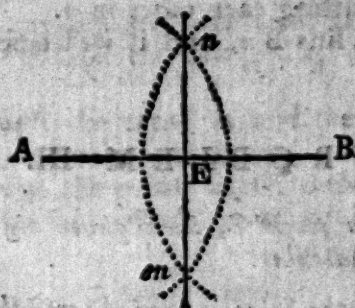
which

which form that angle, the angular point being the center.



PROBLEM I.*

To divide a given line AB into two equal parts.



1. From the points A and B , as centers, with any distance greater than half AB , describe arcs cutting each other in n and m .

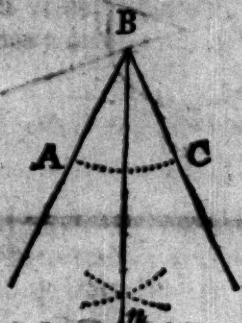
2. Draw the line nm , and the point E , where it cuts AB , will be the middle of the line required.

* The demonstrations of most of these problems may be found in Euclid's Elements.

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PROBLEM II.

To divide a given angle ABC into two equal parts.

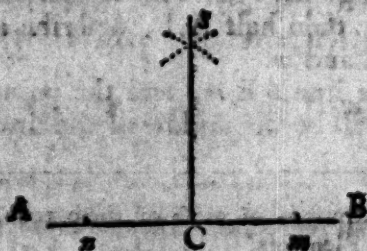


1. From the point B, with any radius, describe the arc A C.
2. From A and C, with one and the same radius, describe arcs cutting each other in n .
3. Draw the line B n , and it will bisect the angle as required.

PROBLEM III.

From a given point C, in a given right line AB, to erect a perpendicular.

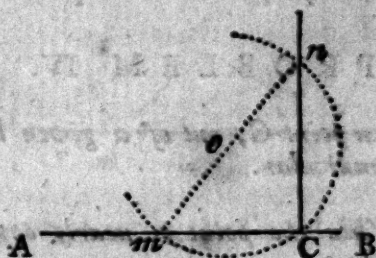
CASE I. When the point is near the middle of the line.



1. On

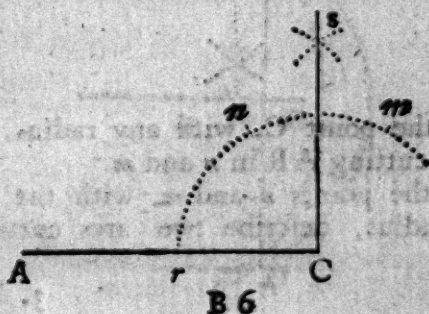
1. On each side of the point C take any two equal distances, Cn and Cm .
2. From n and m , with any radius, greater than Cn or Cm , describe the two arcs cutting in s .
3. Through the points s , C , draw the line sC , and it will be the perpendicular required.

CASE II. *When the point is at, or near, the end of the line.*



1. Take any point o , and with the radius or distance oC , describe the arc mCn , cutting AB in m and C .
2. Through the center o , and the point m , draw the line mon , cutting the arc mCn in n .
3. Through the points n , C , draw the line nC , and it will be the perpendicular required.

Another method.



1. From

12 PRACTICAL GEOMETRY.

1. From the point C , with any radius, describe the arc rm , cutting the line AC in r .

2. From the point r , with the same radius, cross the arc in n ; and from the point n , in like manner, cross it in m .

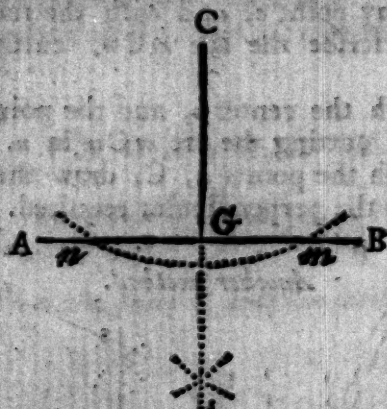
3. From the points n and m , with the same, or any other radius, describe two arcs cutting each other in s .

4. Through the points s and C , draw the line SC , and it will be the perpendicular required.

PROBLEM IV.

From a given point C , out of a given line AB , to let fall a perpendicular.

CASE I. When the point is nearly opposite to the middle of the line.



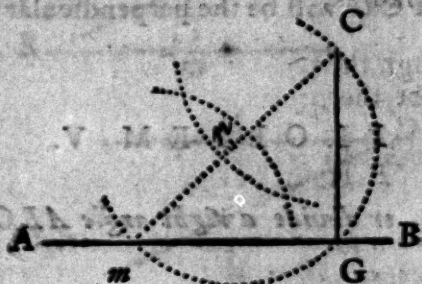
1. From the point C , with any radius, describe the arc mn , cutting AB in n and m .

2. From the points n and m , with the same, or any other radius, describe two arcs cutting each other in s .

3. Through

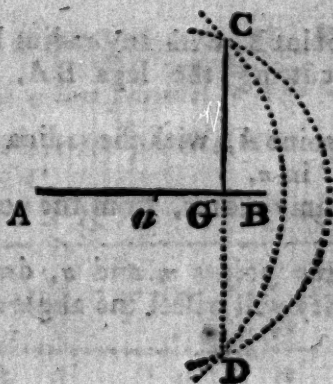
3. Through the points C and s , draw the line CGs , and CG will be the perpendicular required.

CASE II. *When the point is nearly opposite to the end of the line.*



1. To any point m , in the line AB , draw the line Cm .
2. Divide the line Cm into two equal parts in n .
3. From the point n , with the radius nm or nC , describe the arc CGm , cutting AB in G .
4. Through the points C and G , draw the line CG , and it will be the perpendicular required.

Another method.



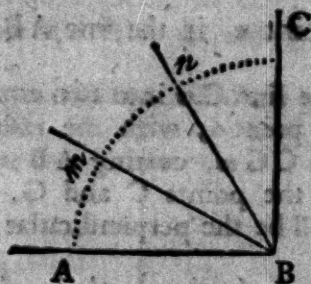
1. From

14 PRACTICAL GEOMETRY.

1. From A, or any other point in AB, with the radius AC, describe the arc CD.
2. From any other point n , in AB, with the radius nC , describe another arc cutting the former in D.
3. Through the points C and D draw the line CGD, and CG will be the perpendicular required.

PROBLEM V.

To trisect, or divide a right angle ABC into three equal parts.



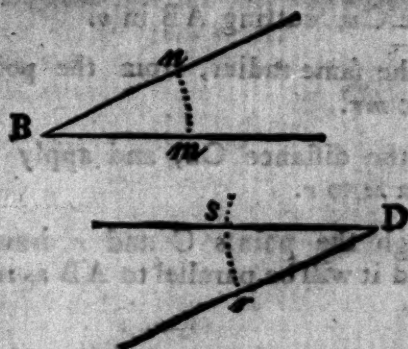
1. From the point B, with any radius BA, describe the arc AC, cutting the legs BA, BC, in A and C.
2. From the point A, with the radius AB or BC, cross the arc AC in n .
3. With the same radius, from the point C, cross it in m .
4. Through the points m and n , draw the lines Bm, Bn, and they will trisect the angle as required.

PRO-

PRACTICAL GEOMETRY. 15

PROBLEM VI.

At a given point D to make an angle equal to a given angle ABC.

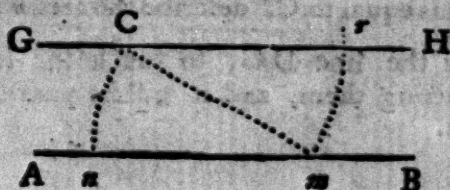


1. From the point B, with any radius, describe the arc nm , cutting the legs Bn and Bm in the points n and m .
2. Draw the line Dr , and from the point D, with the same radius as before, describe the arc rs .
3. Take the distance mn , and apply it to the arc rs , from r to s .
4. Through the points D and s draw the line Ds , and the angle rDs will be equal to the angle mBn as required.

PROBLEM VII.

To draw a line parallel to a given line AB.

CASE I. When the parallel line is to pass through a given point C.

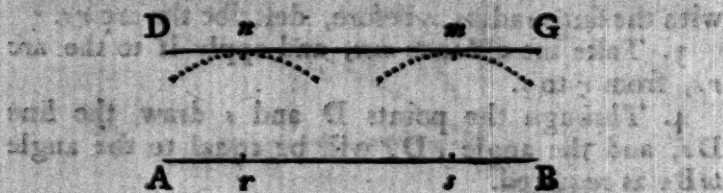


1. From

16 PRACTICAL GEOMETRY.

1. From the point C , to the line AB , draw any right line Cm .
2. From the point m , with the radius mC , describe the arc Cn , cutting AB in n .
3. With the same radius, from the point C , describe the arc mr .
4. Take the distance Cn , and apply it to the arc mr , from m to r .
5. Through the points C and r draw the line $GCrH$, and it will be parallel to AB as required.

CASE II. *When the parallel line is to be at a given distance C from AB .*

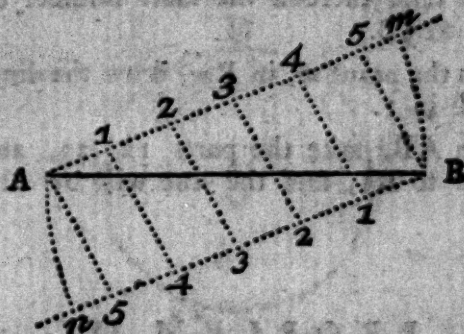


1. From any two points r and s , in the line AB , with a radius equal to C , describe the arcs n and m .
2. Draw the line DG , to touch the said arcs, without cutting them, and it will be parallel to AB as required.

P R O

PROBLEM VIII.

To divide a given line AB into any proposed number of equal parts.

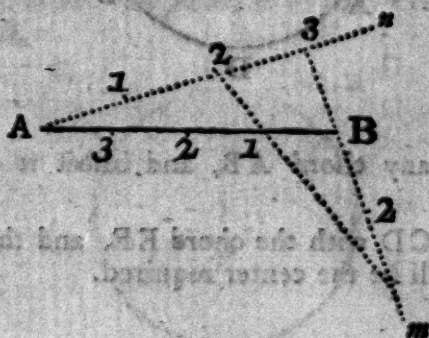


1. From A , one end of the line, draw Am , to make any angle with AB ; and from B , the other end, draw Bn , making an equal angle with the said line.

2. In each of the lines Am , Bn , beginning at A and B , set off as many equal parts, of any length, as AB is to be divided into.

3. Join the points $A5$, $B5$, $A4$, $B4$, $A3$, $B3$, &c. and AB will be divided as required.

Another method.



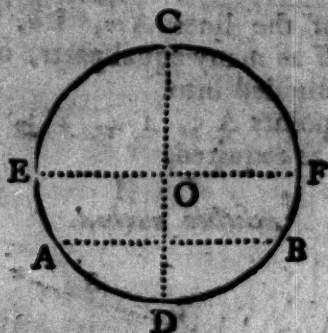
1. From

18 PRACTICAL GEOMETRY.

1. From the point A draw any line Am , and set thereon as many equal parts, wanting one, as AB is to be divided into.
2. Through the points 3 and B , draw the line $3Bm$, and take thereon the same number of parts, each equal to $3B$.
3. From the point m , in Bm , draw the line $m12$, cutting AB in 1 .
4. Upon AB , take the parts 12 , 23 , and $3A$, each equal to $B1$, and the line will be divided as required.

PROBLEM IX.

To find the center of a circle.

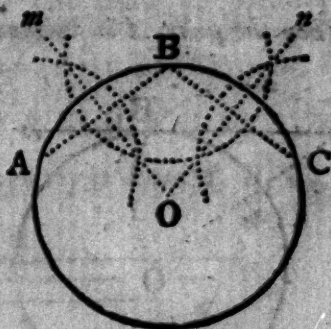


1. Draw any chord AB , and bisect it with the chord CD .
2. Bisect CD with the chord EF , and their intersection O will be the center required.

PRO.

PROBLEM X.

To describe the circumference of a circle through any three given points, *A*, *B*, *C*, provided they are not in a straight line.

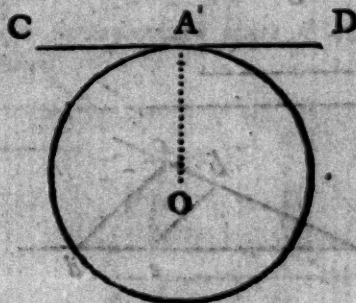


1. From the middle point *B*, draw the chords *BA*, and *BC*.
2. Bisect these chords with the perpendicular lines *mO* and *nO*.
3. From the point of intersection *O*, with the distance *OA*, *OB*, or *OC*, describe the circle *ABC*, and it will be that required.

PROBLEM XI.

To draw a tangent to a given circle, that shall pass through a given point *A*.

CASE I. When the point *A* is in the circumference of the circle.

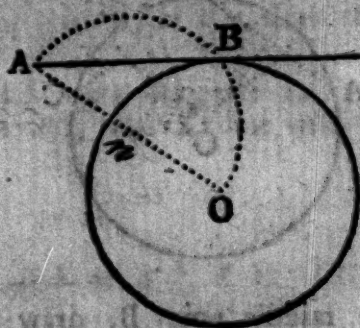


1. From

20 PRACTICAL GEOMETRY.

1. From the center O draw the radius OA .
2. Through the point A draw CD perpendicular to OA , and it will be the tangent required.

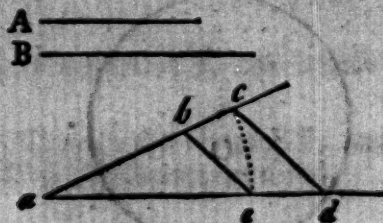
CASE II. *When the point A is without the circle.*



1. From the center O draw OA , and bisect it in n .
2. From the point n , with the radius nA , or nO , describe the semi-circle ABO , cutting the given circle in B .
3. Through the points A and B , draw the line AB , and it will be the tangent required.

PROBLEM XII.

To two given right lines A and B , to find a third proportional.

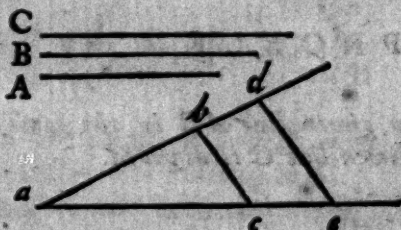


1. From

1. From the point *a* draw two right lines making any angle whatever.
 2. In these lines take *ab* equal to the first term *A*, and *ac*, *ae*, each equal to the second term *B*.
 3. Join *be*, and draw *cd* parallel thereto; then will *ad* be the third proportional required.
- That is $ab (A) : ac (B) :: ae (B) : ad$.

PROBLEM XIII.

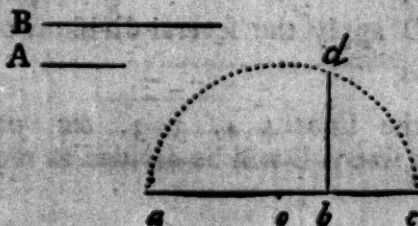
To three given right lines A, B, C, to find a fourth proportional.



1. From the point *a* draw two right lines, making any angle whatever.
 2. In these lines take *ab* equal to the first term *A*, *ac* equal to the second *B*, and *ad* equal to the third *C*.
 3. Join *bc*, and draw *de* parallel thereto, and *ae* will be the fourth proportional required.
- That is $ab (A) : ac (B) :: ad (C) : ae$.

PROBLEM XIV.

Between two given right lines A and B, to find a mean proportional.



1. Draw

22 PRACTICAL GEOMETRY.

1. Draw any right line, in which take cb equal to A , and ba equal to B .

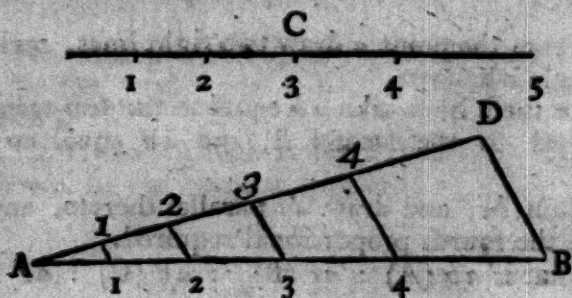
2. Bisect ac in o , and with oa or oc , as radius, describe the semi-circle adc .

3. From the point b draw bd perpendicular to ac , and it will be the mean proportional required.

That is bc (A) : bd :: bd : ba (B).

PROBLEM XV.

To divide a given line AB in the same proportion that another given line C is divided.



1. From the point A draw AD equal to the given line C , and making any angle with AB .

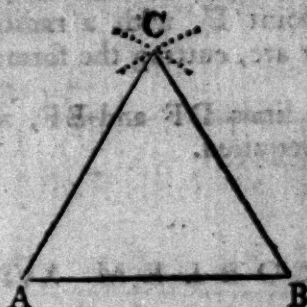
2. To AD apply the several divisions of C , and join DB .

3. Draw the lines 4 4, 3 3, &c. parallel to DB , and the line AB will be divided as required.

PRO.

PROBLEM XVI.

Upon a given right line AB to make an equilateral triangle.



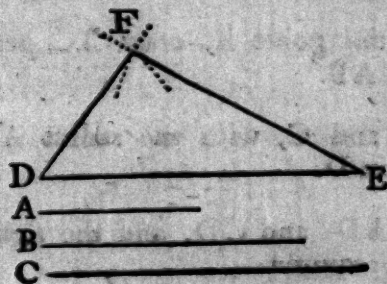
1. From the points A and B, with a radius equal to AB, describe arcs cutting in C.

2. Draw AC and BC, and the figure ACB is the triangle required.

Note. An isosceles triangle may be formed in the same manner.

PROBLEM XVII.

To make a triangle whose three sides shall be respectively equal to three given lines A, B, C, provided any two of them be greater than the third.



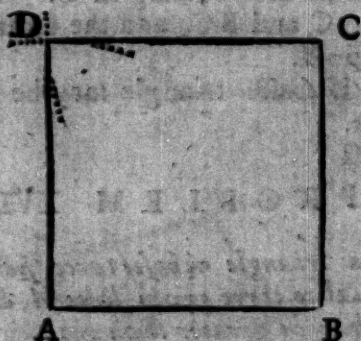
1. Draw

24 PRACTICAL GEOMETRY.

1. Draw a line DE equal to the line C.
2. On the point E, with a radius equal to B, describe an arc.
3. On the point D, with a radius equal to A, describe another arc, cutting the former in F.
4. Draw the lines DF and EF, and DFE will be the triangle required.

P R O B L E M XVIII.

Upon a given line AB to describe a square.

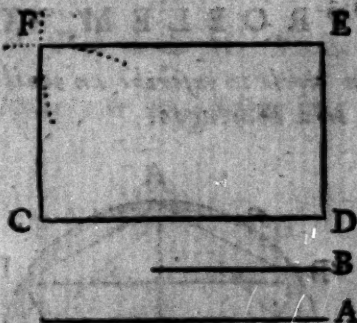


1. From the point B, draw BC perpendicular, and equal to AB.
2. On A and C, with the radius AB, describe arcs cutting in D.
3. Draw AD, and CD, and the figure ABCD is the square required.

P R O-

PROBLEM XIX.

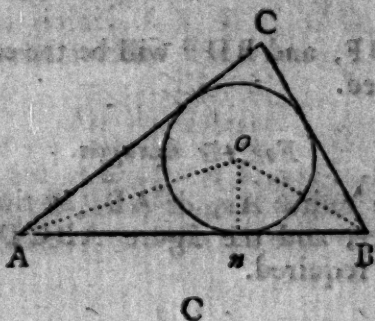
To describe a rectangle, or parallelogram, whose length and breadth shall be equal to two given lines *A* and *B*.



1. Draw the line *CD* equal to *A*, and make *DE* perpendicular thereto, and equal to *B*.
2. On the points *E* and *C*, with the radii *A* and *B*, describe arcs cutting in *F*.
3. Join *C*, *F*, and *E*, *F*, and *CDEF* will be the rectangle required.

PROBLEM XX.

In a given triangle *ABC* to inscribe a circle.

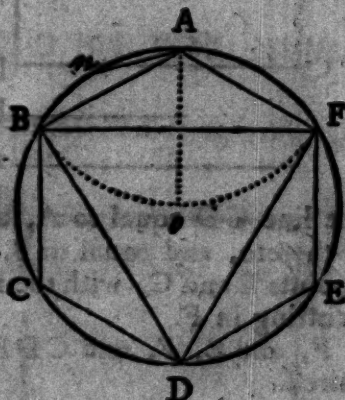


1. Bisect

1. Bisect the angles A and B with the lines AO and BO.
2. From the point of intersection O let fall the perpendicular ON , and it will be the radius of the circle required.

PROBLEM XXI.

In a given circle to inscribe an equilateral triangle, an hexagon, or a dodecagon.



For the equilateral triangle.

1. From any point A as a center, with a distance equal to the radius AO , describe the arc F_0B .
2. Draw the line BF , and make BD equal to BF .
3. Join DF , and BDF will be the equilateral triangle required.

For the hexagon.

Carry the radius A_o, or AB, six times round the circumference, and the figure ABCDEF will be the hexagon required.

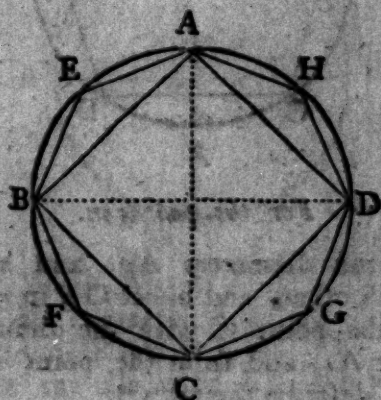
For

For the dodecagon.

Bisect the arc AB of the hexagon in the point π , and the line A π being carried twelve times round the circumference, will form the dodecagon required.

PROBLEM XXII.

To describe a square, or an octagon in a given circle.



For the square.

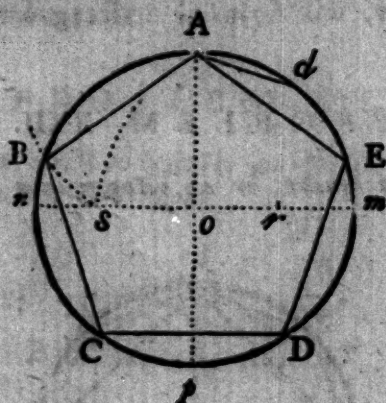
1. Draw the diameters BD and AC, intersecting each other at right angles.
2. Join the points AB, BC, CD, and DA, and ABCD will be the square required.

For the octagon.

Bisect the arc AB of the square in the point E, and the line AE being carried eight times round the circumference, will form the octagon required.

PROBLEM XXIII.*

To inscribe a pentagon, or decagon, in a given circle.



For the pentagon.

1. Draw the diameters As , and nm at right angles to each other, and bisect Om in r .
2. From the point r , with the distance rA , describe the arc As , and from the point A , with the distance As , describe the arc sB .
3. Join the points A and B , and carry the line AB five times round the circle, and it will form the pentagon required.

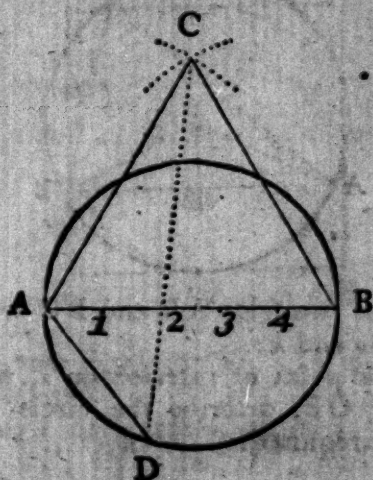
For the decagon.

Bisect the arc AE of the pentagon in d , and the line $A d$ carried ten times round the circumference will form the decagon required.

• Besides the figures here constructed, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method *purely geometrical*.

PROBLEM XXIV.*

In a given circle to inscribe any regular polygon.



1. Draw the diameter AB, on which make the equilateral triangle ACB.

2. Divide the diameter AB into as many equal parts as the required polygon has sides.

3. From the point C, through the second division of the diameter, draw the line CD.

4. Join the points A and D, and the line AD will be the side of the polygon required.

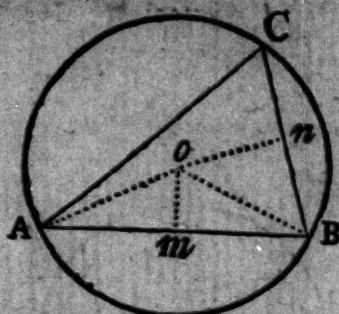
Note. In this construction AD is the side of a pentagon.

* This construction is the invention of Renaldinus, and was first given in his 2d Book *De Refol. &c. Comp. Mathem.* page 367. The rule for polygons in general is only an approximation, but holds true in the equilateral triangle and hexagon.

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PROBLEM XXV.

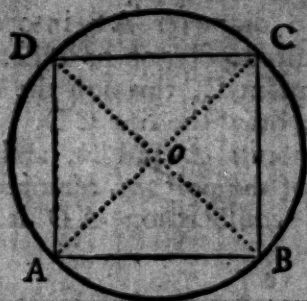
About a given triangle ABC to circumscribe a circle.



1. Bisect the two sides AB and BC with the perpendiculars m and n .
2. From the point of intersection o , with the distance OA or OB, describe the circle ACB, and it will be that required.

PROBLEM XXVI.

About a given square ABCD to circumscribe a circle.

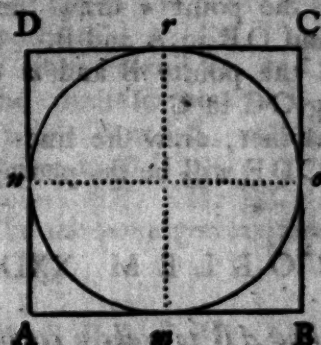


1. Draw the two diagonals AC and BD, intersecting each other in O.
2. From the point O, with the distance OA or OB, describe the circle ABCD, and it will be that required.

PRO-

PROBLEM XXVII.

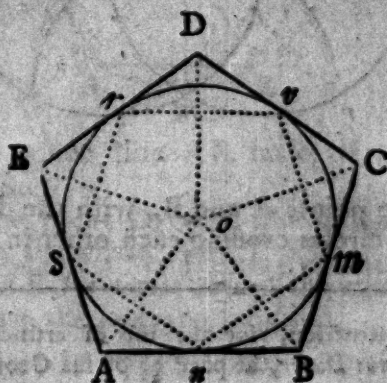
To circumscribe a square about a given circle.



1. Draw any diameter no , and another rm perpendicular thereto.
2. Through the points n, m, o, r , draw the tangents AB, BC, CD , and DA , and $ABCD$ will be the square required.

PROBLEM XXVIII.

About a given circle to circumscribe a pentagon.



C 4

1. Inscribe

1. Inscribe the pentagon *nmors*, and through the middle of each side draw the lines *OA*, *OB*, *OC*, *OD*, and *OE*.

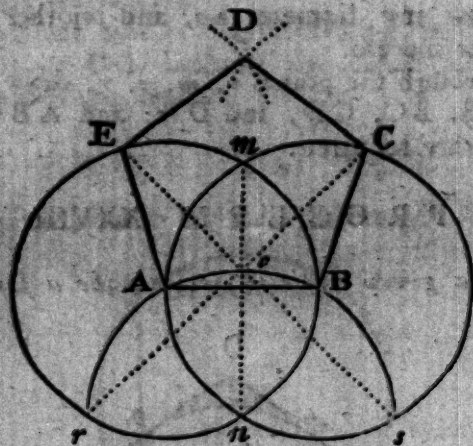
2. Through the point *n* draw the tangent *AB*, meeting *OA* and *OB* in *A* and *B*.

3. Through the points *B* and *m* draw the line *BmC*, meeting *OC* in *C*.

4. In like manner, draw the lines *CD*, *DE* and *EA*, and *ABCDE* will be the pentagon required.

PROBLEM XXIX. *

On a given line AB to make a regular pentagon.



1. On the points *A* and *B*, with the distance *AB*, describe two circles crossing each other in *m* and *n*.

* This construction, for the use of artificers, was first given by *Albertus Durer*, at page 55 of his *Geometry*, printed anno 1532.

2. Draw

2. Draw the line mn , and from the point n , with the same radius as before, describe the arc $rAoS$, cutting the two circles in the points r and s , and the line mn in the point o .

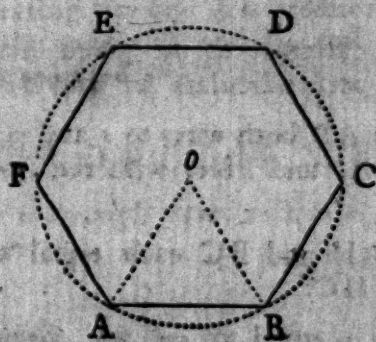
3. Draw the lines ro and so , and produce them till they meet the circumferences in E and C .

4. From the points E and C , with the radius AB , describe arcs crossing in D .

5. Join the points AE , ED , DC , and CB , and $AEDCB$ will be the pentagon required.

PROBLEM XXX.

On a given line AB to make a regular hexagon.



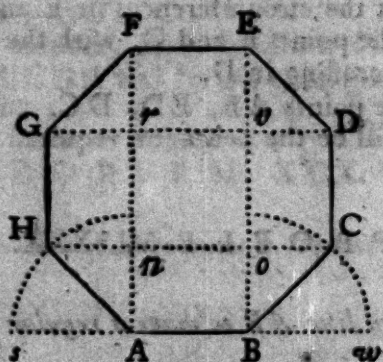
1. On the given line AB make the equilateral triangle AOB .

2. From the point O , with the distance OA or OB , describe the circle $ABCDEF$.

3. Carry AB six times round the circumference, and it will form the hexagon required.

PROBLEM XXXI.

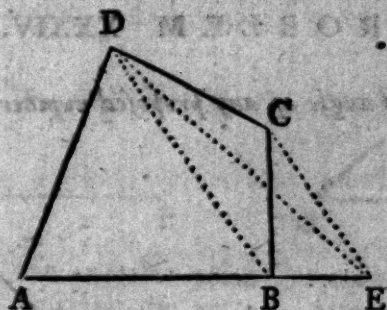
On a given line AB to form a regular octagon.



1. On the extremes of the given line AB erect the indefinite perpendiculars AF and BE.
2. Produce AB both ways to s and w, and bisect the angles, $\angle sA/$ and $\angle oBw$ with the lines AH and BC.
3. Make AH and BC each equal to AB, and draw the line HC.
4. Make ov equal to on , and through v draw GD parallel to HC.
5. Draw HG and CD parallel to AF and BE, and make ve equal to vd .
6. Through E draw EF parallel to AB, and join the points GF and DE, and ABCDEFGH will be the octagon required.

PROBLEM XXXII.

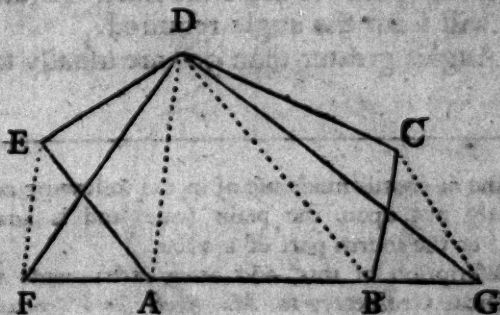
To make a triangle equal to a given trapezium ABCD.



1. Draw the diagonal DB, and make CE parallel thereto, meeting the side AB produced in E.
2. Join the points D and E, and ADE will be the triangle required.

PROBLEM XXXIII.

To make a triangle equal to any right lined figure ABCDEA.



1. Produce the side AB both ways at pleasure.
2. Draw the diagonals DA and DB, and parallel thereto the lines EF and CG.

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3. Join the points DF and DG, and DFG will be the triangle required.

And in nearly the same manner may any right lined figure whatever be reduced to a triangle.

PROBLEM XXXIV.*

To make an angle of any proposed number of degrees.



1. Take the first 60 degrees from the scale of chords, and from the point A, with this radius, describe the arc nm .

2. Take the chord of the proposed number of degrees from the same scale, and apply it from n to m .

3. From the point A draw the lines An and Am , and they will form the angle required.

Note. Angles greater than 90° are usually taken at twice.

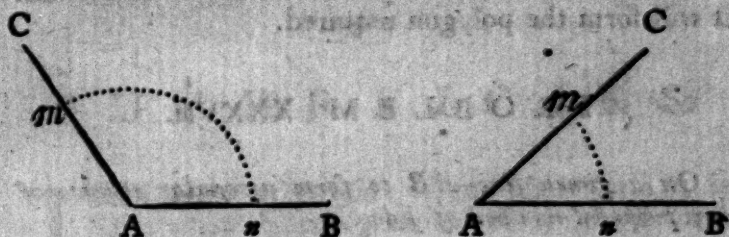
* The line of chords made use of in the following problems, is commonly put upon the plain scale, and is adapted to 90 degrees, or the fourth part of a circle.

For a description of this, and other instruments made use of in Practical Geometry, see *Mr. Robertson's Treatise on such mathematical instruments as are usually put into a portable case.*

P R O.

PROBLEM XXXV.

Any angle BAC being given, to find the number of degrees it contains.



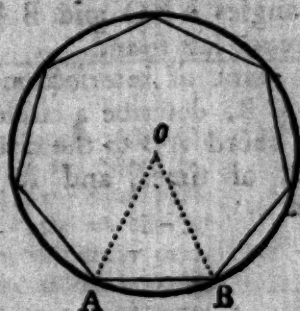
1. From the angular point A, with the chord of 60 degrees, describe the arc nm , cutting the legs in the points n and m .

2. Take the distance nm , and apply it to the scale of chords, and it shews the degrees required.

Note. When the distance nm is greater than 90° , it must be taken at twice, as before.

PROBLEM XXXVI.

In a given circle to inscribe a polygon of any proposed number of sides.



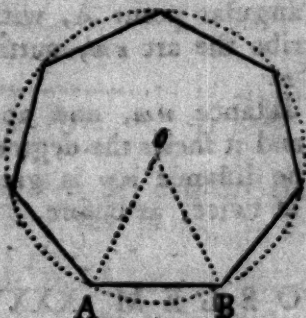
1. Divide

1. Divide 360° by the number of sides, and make an angle $A \odot B$, at the center, whose measure shall be equal to the degrees in the quotient.

2. Join the points A and B, and apply the chord AB to the circumference the given number of times, and it will form the polygon required.

PROBLEM XXXVII.

On a given line AB to form a regular polygon of any proposed number of sides.



1. Divide 360° by the number of sides, and subtract the quotient from 180 degrees.

2. Make the angles ABO and BAO each equal to half the difference last found.

3. From the point of intersection O , with the distance OA or OB , describe a circle.

4. Apply the chord AB to the circumference the proposed number of times, and it will form the polygon required.

OF THE
MENSURATION
OF
SUPERFICIES.

THE *area*, or *measure* of any plane surface, is the whole space contained within the bounds of that surface, without any regard to thickness.

A square whose side is one inch, one foot, or one yard, &c. is called the *measuring unit*, and the area or content of any figure is computed by the number of those squares contained in that figure.

PROBLEM I.

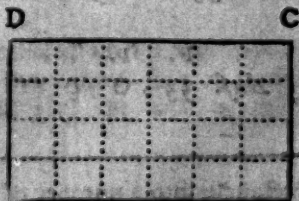
To find the area of a parallelogram; whether it be a square, a rectangle, a rhombus, or a rhomboides.

RULE.*

Multiply the length by the perpendicular height, and the product will be the area.

EXAM-

* Take any rectangle ABCD, and divide each of its sides, respectively, into as many equal parts as is expressed by the number of times they contain the linear measuring unit, and let all the opposite



points of division be connected by A right lines. Then, it is evident that, these lines divide the rectangle

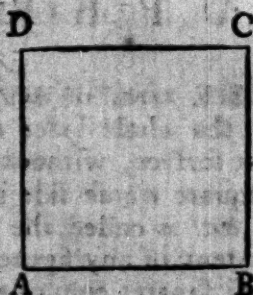
MENSURATION

EXAMPLES.

1. Required the area of the square ABCD whose side is 5 feet 9 inches.

By Decimals.

$$\begin{array}{r}
 \text{F.} \\
 5.75 \\
 5.75 \\
 \hline
 2875 \\
 4025 \\
 2875 \\
 \hline
 33.0625 \\
 12 \\
 \hline
 0.7500 \\
 12 \\
 \hline
 9.0000
 \end{array}$$



By Cross Multiplication.

$$\begin{array}{r}
 \text{F. I.} \\
 5 \ 5 \\
 5 \ 5 \\
 \hline
 28 \ 9 \\
 4 \ 3 \ 9 \\
 \hline
 33 \ 0 \ 9
 \end{array}$$

Ans. F. I. "
33 0 9

By Practice.

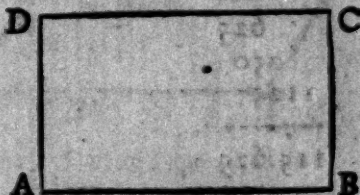
$$\begin{array}{r}
 \text{F. I.} \\
 5 \ 9 \\
 5 \\
 \hline
 28 \ 9 \\
 \text{in.} \\
 6 \text{—}\frac{1}{2} \\
 3 \text{—}\frac{1}{2} \\
 \hline
 2 \ 10 \ 6 \\
 1 \ 5 \ 3 \\
 \hline
 33 \ 0 \ 9
 \end{array}$$

2. Required

rectangle into a number of squares each equal to the superficial measuring unit, and that the number of these squares, or the area of the figure, is equal to the number of linear measuring units.

2. Required the area of the rectangle ABCD, whose length is 13.75 chains, and breadth 9.5 chains.

$$\begin{array}{r}
 13.75 \\
 9.5 \\
 \hline
 6875 \\
 12375 \\
 \hline
 10)130.625 \\
 \hline
 13.0625 \\
 4 \\
 \hline
 0.2500 \\
 40 \\
 \hline
 10.0000
 \end{array}$$



A. R. P.
Ans. 13 0 10

3. Required the area of the rhombus ABCD, whose length AB is 12 6, and its height DE 9 3.

units in the length as often repeated as there are linear measuring units in the breadth or height, that is equal to the length multiplied by the height, *which is the rule.*

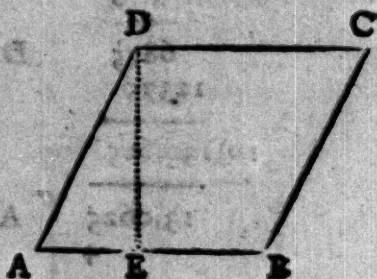
And since a rectangle is equal to an oblique parallelogram standing upon the same base, and between the same parallels, (Euc. I. 35), therefore the rule is true for any parallelogram in general. Q. E. D.

RULE II. If any two sides of a parallelogram be multiplied together, and the product again by the natural sine of their included angle, the last product will give the area of the triangle.

By

By Decimals.

$$\begin{array}{r}
 \text{P.} \\
 12.5 \\
 9.25 \\
 \hline
 625 \\
 250 \\
 1125 \\
 \hline
 115.625 \\
 12 \\
 \hline
 7.500 \\
 12 \\
 \hline
 6.000
 \end{array}$$

*By Cross Multiplication.*

$$\begin{array}{r}
 \text{P.} \quad \text{I.} \\
 12 \quad 6 \\
 9 \quad 3 \\
 \hline
 112 \quad 6 \\
 3 \quad 1 \quad 6 \\
 \hline
 115 \quad 7 \quad 6
 \end{array}$$

By Practice.

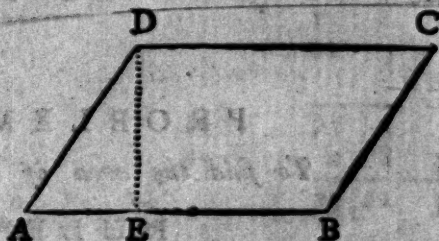
$$\begin{array}{r}
 \text{P.} \quad \text{I.} \\
 12 \quad 6 \\
 9 \\
 \hline
 112 \quad 6 \\
 3 \quad 1 \quad 6 \\
 \hline
 115 \quad 7 \quad 6
 \end{array}$$

in. $3\frac{1}{4}$ *P. I.**Ans.* 115 7 6

4. What is the area of the rhomboides ABCD, whose length AB is 10.52 chains, and height DE 7.63 chains?

10.52

$$\begin{array}{r}
 10.52 \\
 7.63 \\
 \hline
 3156 \\
 6312 \\
 7364 \\
 \hline
 10)80.2676 \\
 \hline
 8.02676 \\
 \hline
 4 \\
 \hline
 .10704 \\
 40 \\
 \hline
 4.28160
 \end{array}$$



Ans. 8 0 4

5. What is the area of a square whose side is 35.25 chains?

ar. ro. po.

Ans. 124 1 1

6. What is the area of a square whose side is 8 feet 4 inches?

ft. in. "

Ans. 69 5 4

7. What is the area of a rectangle whose length is 14 feet 6 inches, and breadth 4 feet 9 inches?

ft. in. "

Ans. 68 10 6

8. Required the area of a rhombus, the length of whose side is 12.24 feet, and height 9.16 feet.

ft. in. "

Ans. 112 5 0496

9. Required the area of a rhomboides whose length is 10.51 chains, and breadth 4.28 chains.

ar. ro. po.

Ans. 4 1 39

10. What



10. What is the area of a rhomboides whose length is 7 feet 9 inches, and height 3 feet 6 inches?

fe. in.
Ans. 27 1 6

PROBLEM II.

To find the area of a triangle.

RULE.*

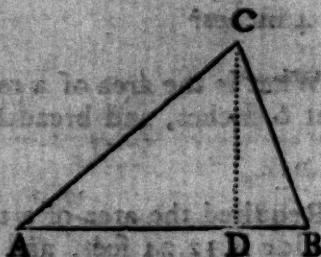
Multiply the base by the perpendicular height, and half the product will be the area.

EXAMPLES.

1. Required the area of the triangle ABC, whose base AB is 10 feet 9 inches, and height DC 7 feet 3 inches.

By Decimals.

f.
10.75
7.25
—
5375
2150
7525
—
77.9375
12
—
11.2500
12
—
3.0000



* A triangle is half a parallelogram of the same base and altitude, (Euc. I. 41.) and therefore the truth of the rule is evident.

By

By Cross Multiplication.

$$\begin{array}{r}
 \text{F. I.} \\
 10 \ 9 \\
 \underline{7 \ 3} \\
 75 \ 3 \\
 \underline{2 \ 8 \ 3} \\
 77 \ 11 \ 3
 \end{array}$$

Ans. $\begin{array}{r} \text{F. I.} \\ 77 \ 11 \ 3 \end{array}$

By Practice.

$$\begin{array}{r}
 \text{F. I.} \\
 10 \ 9 \\
 \underline{7} \\
 75 \ 3 \\
 \underline{2 \ 8 \ 3} \\
 77 \ 11 \ 3
 \end{array}$$

in.
 $3 - \frac{1}{3}$

2. What is the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches?

ft. in. "
Ans. 216 11 4

3. What is the area of a triangle whose base is 16.75 feet, and height 6.25 feet?

ft. in. "
Ans. 104 8 3

4. Required the area of a triangle whose base is 12.25 chains, and height 8.5 chains.

ar. ro. po.
Ans. 10 1 26

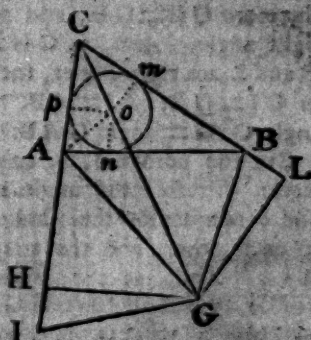
PROBLEM III.

To find the area of a triangle whose three sides only are given.*

RULE.

* *Demon.* Take any triangle ABC, and let ump be its inscribed circle, whose center is o ; join Ao , and Co , and let fall the perpendiculars on , om , and op ; in CA produced take $AH = Bn$, and erect the perpendicular HG , meeting Co produced in G ; make BL and HI each equal to An , and join GL , GI , GB , and GA .

Now, as $Cp = Cm$, $Ap = An$, and $Bm = Bn$, it is evident that



CH,

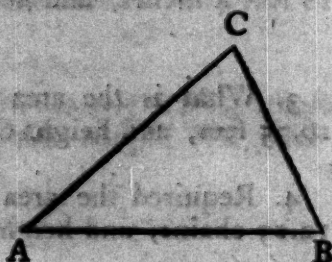
RULE

From half the sum of the three sides subtract each side severally; multiply the half sum and three remainders continually together, and the square root of the product will be the area required.

EXAMPLES.

1. Required the area of the triangle ABC, whose three sides BC, CA and AB, are 24, 36, and 48 chains respectively.

$$\begin{array}{r}
 24 \\
 36 \\
 48 \\
 \hline
 2)108 \text{ sum} \\
 \hline
 54 \text{ half-sum}
 \end{array}$$



54

CH, or CL, will be half the perimeter of the triangle, and that HA, Ap, and pC will be the differences between the half perimeter and each side respectively. And since CH = CL, CG common, and the angle HCG = angle LCG, therefore GL = GH, and the angle GLC = angle GHC = a right angle. Also, as GH = GL, HI = LB, and the angles H and L are right angles, therefore GI = GB. In like manner, as GI = GB, AI = AB, and GA common, therefore the angle GAI = angle GAB.

But the points A, n, o, p, fall in the circumference of a circle, therefore the angle IAB = angle pon (Euc. III. 22.) and consequently their halves HAG and Aop are also equal to each other, and the triangle AHG similar to the triangle Aop. And, as the triangles Cpo and CHG are also similar, we shall have HG : po :: HC : Cp and pA : HG :: po : AH; whence HG x pA : po x HG :: HC x po : Cp x AH or pA : po :: HC x po : Cp x AH, or pA x CH : po x CH :: HC x po : Cp x AH, which is the same as the

54	54	54
24	36	48
30 <i>first diff.</i>	18 <i>second diff.</i>	6 <i>third diff.</i>

30	
540	
6	
3240	
54	
12960	
16200	
174960	(418.282 = area required
16	

81)149
81

328)6860
6624

8362)23600
16724

83648)687600
669184

836562)1841600
1673124
168476

2. Required

the rule; for if this be expressed algebraically, it will be
 $\sqrt{CH \times pA \times Cp \times AH} = \sqrt{CH^2 \times po^2} = CH \times po$, which is
 the rule. Q. E. D. Cor. 1.

2. Required the area of a triangle whose three sides are 13, 14 and 15 feet. *Ans.* 84 square feet.

3. How many acres are there in a triangle whose three sides are 49.00, 50.25 and 25.69 chains?

Ans. 61 ac. 3 ro. 36.8064 po.

4. Required the area of a right angled triangle, whose hypotenuse is 50, and the other two sides 30 and 40. *Ans.* 600.

5. Required the area of an equilateral triangle whose side is 25. *Ans.* 270.625

PROBLEM IV.

Any two sides of a right angled triangle being given to find the third side.

RULE.*

1. *When the two legs are given, to find the hypotenuse.*

Add the square of one of the legs to the square of the other, and the square root of the sum will be equal to the hypotenuse.

Cor. 1. If the triangle be right angled, the rectangle of the half perimeter, and the difference between the half perimeter and the hypotenuse will be the area; because when C A B is a right angle, B L will be equal to p o.

Cor. 2. If the triangle be equilateral, $\frac{1}{4}\sqrt{3}$ multiplied by the square of the side will be the area; because, in that case, the perimeter is three times the side, and the three differences are all equal to each other.

RULE II. Any two sides of a triangle being multiplied together, and the product again by half the natural sine of their included angle, will give the area of the triangle.

* By Euc. 47. I. $AB^2 + BC^2 = AC^2$, or $AC^2 - AB^2 = BC^2$, and therefore $\sqrt{AB^2 + BC^2} = AC$, or $\sqrt{AC^2 - AB^2} = BC$, and is the same as the rule.

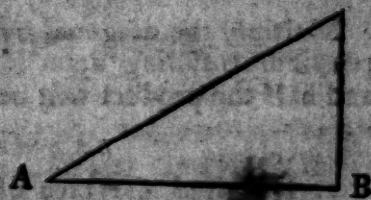
2. *When*

2. When the hypotenuse and one of the legs are given, to find the other leg.

From the square of the hypotenuse take the square of the given leg, and the square root of the remainder will be equal to the other leg required.

EXAMPLES.

1. In the right angled triangle ABC; the base AB is 56, and the perpendicular BC 33: what is the hypotenuse?

$\begin{array}{r} 33 \\ 33 \\ \hline 99 \\ 99 \\ \hline 1089 \end{array}$	$\begin{array}{r} 56 \\ 56 \\ \hline 336 \\ 280 \\ \hline 3136 \end{array}$	
$1089 \text{ square of } BC$	$3136 \text{ square of } AB$	
	$\begin{array}{r} 1089 \\ \hline 4225 \end{array}$	
	$4225 (65 = \text{hypotenuse } AC)$	
	$\begin{array}{r} 36 \\ \hline 125 \overline{)625} \\ 625 \\ \hline \end{array}$	

2. If the hypotenuse AC be 53, and the base AB 45: what is the perpendicular BC?

$\begin{array}{r} 45 \\ 45 \\ \hline 225 \\ 180 \\ \hline 2025 \end{array}$	$\begin{array}{r} 53 \\ 53 \\ \hline 159 \\ 265 \\ \hline 2809 \end{array}$
$2025 \text{ square of } AB$	$2809 \text{ square of } AC$
	$\begin{array}{r} 2025 \\ \hline 784 \end{array}$
	$784 (28 = \text{perpendicular } CB)$
	$\begin{array}{r} 4 \\ \hline 48 \overline{)384} \\ 384 \\ \hline \end{array}$
	D

3. The

3. The base of a right angled triangle is 77, and the perpendicular 36: what is the hypotenuse?

Ans. 85.

4. The hypotenuse of a right angled triangle is 109, and the perpendicular is 60: what is the base?

Ans. 91.

PROBLEM V.

To find the area of a trapezium.

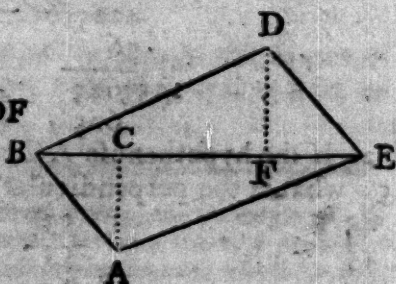
RULE.*

Multiply the diagonal by the sum of the two perpendiculars falling upon it from the opposite angles, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezium BAED, whose diagonal BE is 84, the perpendicular AC 21, and DF 28.

$$\begin{array}{r}
 28 \\
 21 \\
 \hline
 49 \text{ sum of AC \& DF} \\
 84 \text{ diagonal} \\
 \hline
 196 \\
 392 \\
 2) 4116 \\
 \hline
 2058 \text{ the answer.}
 \end{array}$$



2. Required

* *Demon.* The area of the triangle BDE is $= \frac{BE \times DF}{2}$; and the area of the triangle BAE is $= \frac{BE \times AC}{2}$; and therefore the sum of their areas, or the area of the whole trapezium, is $= \frac{BE \times DF}{2} + \frac{BE \times AC}{2} = \frac{DF + AC}{2} \times BE$. Q. E. D.

If

2. Required the area of a trapezium whose diagonal is 80.5, and the two perpendiculars 24.5 and 30.1.

Ans. 2197.65.

3. What is the area of a trapezium whose diagonal is 108 feet 6 inches, and the perpendiculars 56 feet 3 inches, and 60 feet 9 inches? *Ans.* 6347 *ft.* 3 *in.*

PROBLEM VI.

To find the area of a trapezoid, or a quadrangle, two of whose opposite sides are parallel.

RULE.

Multiply the sum of the parallel sides by the perpendicular distance between them, and half the product will be the area.

EXAMPLES.

1. Required the area of the trapezoid ABCD, whose sides AB and DC are 321.51 and 214.24, and perpendicular DE 171.36.

If the trapezium can be inscribed in a circle, that is, if the sum of any two opposite angles is equal to two right angles, or 180° , the area may be found thus:

Rule. From half the sum of the four sides subtract each side severally; then multiply the four remainders continually together, and the square root of the product will be the area.

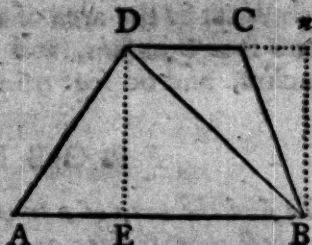
$$\begin{aligned}
 & \text{* Demon. The } \triangle ABD \text{ is } = \frac{AB \times DE}{2}, \text{ and the } \triangle BCD \\
 & = \frac{DC \times B_n}{2}, \text{ or (because } B_n = DE) = \frac{DC \times DE}{2}, \therefore \triangle ABD \\
 & + \triangle BCD, \text{ or the whole trapezoid is, } = \frac{AB \times DE}{2} + \\
 & \frac{DC \times DE}{2} = \frac{AB + DC \times DE}{2}. \quad Q. \quad E. \quad D.
 \end{aligned}$$

D 2

321.51

$$\begin{array}{r}
 321.51 \\
 214.24 \\
 \hline
 535.75 \\
 171.36 \\
 \hline
 321450 \\
 160725 \\
 53575 \\
 375025 \\
 53575 \\
 \hline
 2)91806.1200
 \end{array}$$

45903.0600 the answer.



2. The parallel sides of a trapezoid are 12.41 and 8.22 chains, and the perpendicular distance 5.15 chains; required the area.

ac. ro. po.

Ans. 5 1 9

3. Required the area of a trapezoid whose parallel sides are 25 feet 6 inches and 18 feet 9 inches, and the perpendicular distance 10 feet 5 inches?

fr. in. ~

Ans. 230 5 7

PROBLEM VII.

To find the area of a regular polygon.

RULE.*

Multiply half the perimeter of the figure by the perpendicular falling from its center upon one of the sides, and the product will be the area.

EXAM-

* *Demon.* Every regular polygon is composed of as many equal triangles as it has sides, consequently the area of one of those triangles being multiplied by the number of sides must give

EXAMPLES.

1. Required the area of the regular pentagon ABCDE whose side AB, or BC, &c. is 25 feet, and perpendicular OP 17.2 feet?

$$\begin{array}{r} 25 \\ 5 \\ \hline \end{array}$$

$$2)125$$

$$62.5 = \text{half perimeter}$$

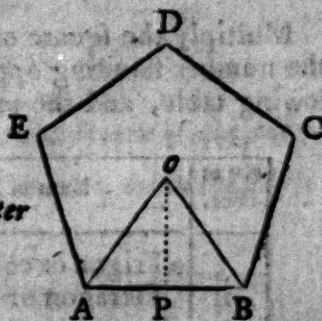
$$17.2$$

$$1250$$

$$4375$$

$$625$$

$$1075.00 = 1075 \text{ square feet, the answer.}$$



2. Required the area of a hexagon whose side is 14.6 feet, and perpendicular 12.64 feet?

Ans. 553.632 square feet.

3. Required the area of a heptagon whose side is 19.38, and perpendicular from the center 20?

Ans. 1356.6.

4. Required the area of an octagon whose side is 9.941, and perpendicular 12?

Ans. 467.168.

give the area of the whole figure; but the area of either of the triangles is equal to the rectangle of the perpendicular and half the base, and therefore the rectangle of the perpendicular and half the sum of the sides is equal to the area of the whole polygon; thus, $OP \times \frac{AB}{2}$ is = area of the $\triangle AOB$, and

$$OP \times \frac{5AB}{2} = \text{area of the polygon ABCDE. } Q. E. D.$$

D 3

P R Q

PROBLEM VIII.

To find the area of a regular polygon when the side only is given.

R U L E. *

Multiply the square of the side of the polygon, by the number standing opposite to its name in the following table, and the product will be the answer.

N ^o of sides.	Names.	Multipliers.
3	Trigon or equil. Δ	0.433013—
4	Tetragon or square	1.000000
5	Pentagon	1.720477+
6	Hexagon	2.598076+
7	Heptagon	3.633912+
8	Octagon	4.828427+
9	Nonagon	6.181824+
10	Decagon	7.694209—
11	Undecagon	9.365641+
12	Duodecagon	11.196152+

EXAM-

* *Demon.* The multipliers in the table are the areas of the polygons to which they belong when the side is unity or 1.

Now, as all regular polygons, of the same number of sides, are similar to each other, and as similar figures are as the squares of their like sides, (*Euc. VI. 20.*) therefore 1st: multiplier in the table :: square of the side of any polygon : area of that polygon; or, which is the same thing, the square of the side of any polygon $\times$ by its tabular number is = area of the polygon. Q. E. D.

The table is formed by trigonometry thus: As radius = 1 : tang. $\angle$ OBP :: BP ($\frac{1}{2}$) : PO = $\frac{BP \times \text{tang. } \angle \text{ OBP}}{\text{radius}} = \frac{1}{2}$
 tang. $\angle$ OBP; then OP $\times$ BP = $\frac{1}{2}$ tang. $\angle$ OBP = area of

EXAMPLES.

1. Required the area of a pentagon whose side is 15.

$$\begin{array}{r} 15 \\ 15 \\ \hline 75 \\ 15 \\ \hline \end{array}$$

225 = square of the side.

1.720477 = multiplier, or area, when the side is 1.
225

$$\begin{array}{r} 8602385 \\ 3440954 \\ 3440954 \\ \hline \end{array}$$

387.107325 = area required.

of the $\triangle AOB$; and $\frac{1}{2}$ tang. OBP $\times$ number of sides = tabular number, or the area of the polygon.

The angle OBP, together with its tangent, for any polygon of not more than 12 sides, is shewn in the following table.

N ^o of sides.	Names.	Angle OBP	Tangents.
3	Trigon	30°	.57735 + = $\frac{1}{3}\sqrt{3}$
4	Tetragon	45°	1.00000 = 1×1
5	Pentagon	54°	1.37638 + = $\sqrt{1 + \frac{2}{5}}\sqrt{5}$
6	Hexagon	60°	1.73205 + = $\sqrt{3}$
7	Heptagon	64° $\frac{2}{7}$	2.07652 +
8	Octagon	67° $\frac{1}{2}$	2.41421 + = $1 + \sqrt{2}$
9	Nonagon	70°	2.74747 +
10	Decagon	72°	3.07768 + = $\sqrt{5 + 2}\sqrt{5}$
11	Undecagon	73° $\frac{7}{11}$	3.40568 +
12	Duodecagon	75°	3.73205 + = $2 + \sqrt{3}$

2. The side of a hexagon is 5 feet 4 inches; what is the area? *Ans.* 73.89

3. Required the area of an octagon whose side is 16? *Ans.* 1236.0773

4. Required the area of a decagon whose side is 20.5? *Ans.* 3233.4913

PROBLEM IX.

The diameter of a circle being given to find the circumference; or, the circumference being given to find the diameter.

RULE I.*

As 7 is to 22 so is the diameter to the circumference. Or, as 22 is to 7 so is the circumference to the diameter.

EXAM-

* The proportion of the diameter of a circle to its circumference has never yet been exactly attained. Nor can a square, or any other right lined figure, be found, that shall be equal to a given circle. This is the famous problem called the *squaring of the circle*, which has exercised the abilities of the greatest mathematicians for ages, and been the occasion of so many endless disputes. Several persons of considerable eminence have, at different times, pretended that they had discovered the exact quadrature; but their errors have soon been detected, and it is now generally looked upon as a thing impossible to be done.

But though the relation between the diameter and circumference cannot be expressed in known measure, it may yet be approximated to any assigned degree of exactness. And thus that incomparable Geometer, the great *Archimedes*, about two thousand years ago, had discovered this proportion to be nearly as 7 is to 22, *which is the same as our first rule.*

By inscribing and circumscribing polygons of 96 sides, he found the proportion to be less than $3\frac{1}{7}$, but greater than $3\frac{1}{8}$
to

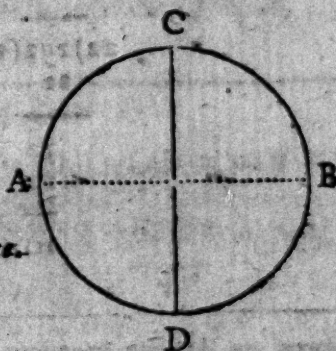
EXAMPLES.

1. If the diameter AB or CD, of a circle be 9, what is the circumference?

$$7 : 22 :: 9$$

$$\begin{array}{r} 9 \\ \hline 7 \overline{)198} \end{array}$$

$28\frac{1}{2}$ the circumference.



2. If

to 1; and from thence inferred the ratio above-mentioned; as may be seen in his book *de dimensione circuli*. And in this manner was the problem more-anciently performed by *Pbilo Gedarenfis*, and by *Apollonius Pergæus*, in a work, not come to our hands, called *Ocyroboos*, as we are informed by *Eutocius* in his commentary on Archimedes.

The proportion of *Pieta* and *Metius* is that of 113 to 355, which is something more exact than the former, and is the same as the second rule.

But the first who ascertained this ratio to any great degree of exactness was *Van Ceulen*, a Dutchman, in his book *de Circolo & Adscriptis*. He found that if the diameter of a circle was 1, the circumference would be 3,141592653589793238462643383279502884 nearly. And this is exactly true to 36 places of decimals, and was effected by means of the continual bisection of an arc of a circle, which was so exceedingly troublesome and laborious, that it must have cost him incredible pains. It is said to have been thought so curious a performance, that the numbers were cut on his tomb-stone, in *St. Peter's Church-Yard, at Leyden*. This last number was not only confirmed, but extended to double the number of

2. If the circumference of a circle BCAD be 36 feet, what is the diameter?

$$22 : 7 :: 36$$

7

22)252($11\frac{10}{11}$ feet, the diameter.

22

32

22

10

RULE

places, by the late ingenious *Mr. Abraham Sharp*, of *Little Horton*, near *Bradford*, in *Yorkshire*.

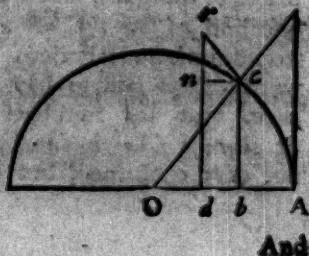
But since the invention of *Fluxions*, and the *Summation of Infinite Series*, there have been several methods found out for doing the same thing with less labour and trouble, and far more expedition. The late *Mr. John Machin*, *Professor of Astronomy in Gresham College*, has, by these means, given a quadrature of the circle which is true to 100 places of decimals; and *M. De Lagny* and *M. Euler* have carried it still farther. And these last expressions are so extremely near the truth, that, except the ratio could be completely obtained, we need not wish for a greater degree of accuracy.

The method of obtaining this proportion from the doctrine of fluxions may be shewn as follows:

Take *Ac* any arc of a circle, and let *cr* be an indefinitely small tangent at the point *c*.

Then draw the lines as in the figure, and put *OA* = *r*, *Ab* = *x*, *bc* = *y*, *AT* = *t*, and *Ac* = *z*. T

Now, since the triangles *xcn*, *cbO*, and *TAO* are similar, we shall have *bc* (*y*) : *co* (*r*) :: *rn* (*x*) : *cr* = *z* = $\frac{rx}{y} = \frac{rx}{\sqrt{2rx - x^2}}$,
= fluxion of the arc, in terms of the versed sine.



And

RULE II.

As 113 is to 355 so is the diameter to the circumference. Or, as 355 is to 113 so is the circumference to the diameter.

EXAM-

And also, $Ob (\sqrt{r^2 - y^2}) : Oc (r) :: nr (\dot{y}) : cr = \dot{z} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}}$ = fluxion of the arc in terms of the sine.

And, in like manner, $AT (t) : OT (\sqrt{r^2 + t^2}) :: en (\dot{x}) : cr = \dot{z} = \frac{\dot{x} \sqrt{r^2 + t^2}}{t}$; but $OT (\sqrt{r^2 + t^2}) : OA (r) ::$

$Oc (r) : Ob = \frac{r^2}{\sqrt{r^2 + t^2}}$, and therefore $Ab (x) = r - \frac{r^2}{\sqrt{r^2 + t^2}}$, whose fluxion is $\frac{r^2 \dot{t}}{(\sqrt{r^2 + t^2})^{\frac{3}{2}}} = \dot{z}$; and consequently $\frac{\dot{x} \sqrt{r^2 + t^2}}{t} = \frac{\sqrt{r^2 + t^2}}{t} \times \frac{r^2 \dot{t}}{(\sqrt{r^2 + t^2})^{\frac{3}{2}}} = \frac{r^2 \dot{t}}{t \sqrt{r^2 + t^2}}$ = fluxion of the arc in terms of the tangent.

Now, from any of the three forms of fluxions here found, their fluents, or the value of the arc itself, will become known.

But the third form, expressed in terms of the tangent, will be the most convenient, because it is entirely free from radical quantities; and therefore, if $\frac{r^2 \dot{t}}{t \sqrt{r^2 + t^2}}$ be converted into an in-

finite series, we shall have $\dot{z} = \frac{r^2 \dot{t}}{r^2 + t^2} = \dot{t} - \frac{r^2 \dot{t}}{r^2} + \frac{r^4 \dot{t}}{r^4} - \frac{r^6 \dot{t}}{r^6} \&c$; and its fluent $z = t - \frac{r^2}{3r^2} + \frac{r^5}{5r^4} - \frac{r^7}{7r^6} + \frac{r^9}{9r^3} \&c$. = length of the arc Ac .

D 6.

Then

EXAMPLES.

1. If the diameter of a circle be 10 feet, what is the circumference?

$$As\ 113 : 355 :: 10$$

113)3550(31 $\frac{47}{113}$, feet, the circumference.

339

160

113

47

2. What

Then by taking $Ac =$ to any given arc whose tangent can be found in terms of the radius, the series will become known; and being repeated as often as Ac is contained in the whole circumference, we shall have the length of the circumference in terms of the diameter.

Thus, if the radius be 1, and Ac be $\frac{1}{8}$ part of the circumference, or 45° , its tangent will be equal to the radius, and the series will become $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$ &c. $=$ arc of 45° , and $8 \times : 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13}$ &c. $=$ whole circumference.

This series is the simplest form that can possibly be obtained, but in order to get another that will converge faster, we must take a smaller arc; as for instance, suppose that of 30° , or $\frac{1}{6}$ part of the circumference.

Then, since the tangent of 30° , to radius 1, is $\sqrt{\frac{1}{3}}$, the general series will become $\sqrt{\frac{1}{3}} \times : 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^3} - \frac{1}{7 \cdot 3^5} + \frac{1}{9 \cdot 3^7}$ &c. $=$ arc of 30° , and $12 \sqrt{\frac{1}{3}} \times : 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^3} - \frac{1}{7 \cdot 3^5} + \frac{1}{9 \cdot 3^7}$ &c. $=$ whole circumference, and so for any other arc whatever.

Those.

2. What is the diameter of a circle whose circumference is 50?

$$\text{As } 355 : 113 :: 50$$

$355 \overline{) 5650} (15 \text{ the diameter.}$

355

2100

1775

325

RULE III.

Multiply the diameter by 3.1416 and the product will be the circumference; or

Divide the circumference by 3.1416 and the quotient will be the diameter.

EXAMPLES.

1. If the diameter of a circle be 17 what is the circumference?

3.1416

17

219912

31416

$53.4072 = \text{circumference.}$

Those who would wish to see the methods of *Machin*, *Euler*, &c. may consult Dr. Hutton's *Mensuration*, and a paper of his in the *Philosophical Transactions* upon this subject.

2. If

2. If the circumference of a circle be 355 what is the diameter?

3.1416)355.000000(112.999 = the diameter.

31416

40840

31416

94240

62832

314080

282744

313360

282744

306160

282744

23416

3. What is the circumference of a circle whose diameter is 40 feet? *Ans.* 125.6640.

4. If the circumference of a circle be 12, what is the diameter? *Ans.* 3.81972.

5. The earth is a globe, and its circumference is known to be 25000.8528 miles, what is its diameter? *Ans.* 7958.

PROBLEM X.

To find the length of any arc of a circle.

RULE

RULE I.

From 8 times the chord of half the arc subtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc *nearly*.

* *Demon.* Let the radius $OD=r$, and sine $DB=s$. Then will the chord $DC = \sqrt{s^2 + BC^2} = \sqrt{s^2 + r - (OB)^2} = \sqrt{s^2 + r - \sqrt{r^2 - s^2}}^2 = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4} \&c.$

And therefore 8 times the chord $DC = 8s + \frac{8s^3}{r^2} + \frac{7s^5}{16r^4} \&c.$

And consequently 8 times the chord DC —chord DE ($2s$)
 $= 6s + \frac{s^3}{r^2} + \frac{7s^5}{16r^4} \&c.$ whose $\frac{1}{3}$ part is $2s + \frac{s^3}{3r^2} + \frac{7s^5}{48r^4} \&c.$

But the length of the arc DC , whose sine is s , is known to be $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} \&c.$ and therefore the arc $DE = 2s + \frac{s^3}{3r^2} + \frac{6s^5}{40r^4} \&c.$ which differs from $2s + \frac{s^3}{6r^2} + \frac{7s^5}{48r^4} \&c.$ only by a small quantity, and shews the rule to be very near the truth. Q. E. D.

Coroll. When the chord of the whole arc is given the rule will be $8 \sqrt{s^2 + r - \sqrt{r^2 - s^2}}^2 - 2s$

A great number of approximating rules might be given for finding the arc of a circle, but the two given in the text, and the three following ones, will be found sufficient.

Rule 1. .01745 &c. $\times$ rad. $\times$ number of degrees in any arc = to the length of that arc.

Rule 2. $4OC \times \sqrt{\frac{3BC}{6OC - BC}} = \text{arc DCE nearly.}$

Rule 3. $10OC \times \sqrt{\frac{5BC}{10OC - 3BC}} + 4 \sqrt{2OC \times BC} \times \frac{\pi}{9} = \text{arc DCE extremely near.}$

EXAMPLE

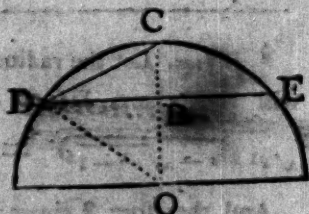
EXAMPLES.

1. The chord of the whole arc DE is 48, and the versed sine BC of half the arc is 18: what is the length of the arc DCE?

$$\begin{array}{r} 18 \\ 18 \\ \hline 144 \\ 18 \\ \hline \end{array}$$

324 = square of BC.

576 = square of DB, or $\frac{1}{2}$ DE.



900 (30 = DC, the chord of $\frac{1}{2}$ the arc.)

$$\begin{array}{r} 240 \\ 48 = DE. \\ \hline \end{array}$$

$$3)192$$

64 = length of the arc required.

2. The chord of the whole arc is 50.8, and the chord of half the arc is 30.6: required the length of the arc. *Ans.* 64.6

3. The length of the whole chord is 6, and the length of the chord of half the arc 3.0438: what is the length of the arc? *Ans.* 6.1268.

RULE II.*

1. Add

* *Demon.* Let the diameter $CF = d$, the versed sine $CB = v$, and the chord $DE = c$.

Then.

1. Add the square of half the chord to the square of the versed sine of half the arc, and this sum divided by the versed sine will give the diameter.

2. Take $\frac{4}{3}$ of the versed sine from the diameter, and divide $\frac{2}{3}$ of the versed sine by the remainder.

3. To the quotient, last found, add 1; and this sum multiplied by the chord of the whole arc, will give the length of the arc *nearly*.

EXAMPLES.

1. If the chord DE be 48, and the versed sine CB 18: what is the length of the arc?

Then will the rule be expressed by $\frac{\frac{2}{3}v}{d - \frac{4}{3}v} + 1 \times c$, or by

$$\frac{\frac{2}{3}v}{d - \frac{4}{3}v} + 1 \times 2\sqrt{dv - v^2}.$$

But $\frac{\frac{2}{3}v}{d - \frac{4}{3}v} + 1 = 1 + \frac{2v}{3d} + \frac{41v^2}{75d^2} + \frac{41 \times 41v^3}{75 \times 50d^3} \&c.$ and

$$2\sqrt{dv - v^2} = 2\sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{8d^2} \&c. \text{ Whence } \frac{\frac{2}{3}v}{d - \frac{4}{3}v}$$

$$+ 1 \times 2\sqrt{dv - v^2} = 1 + \frac{2v}{3d} + \frac{41v^2}{75d^2} \&c. \times 2\sqrt{dv} \times 1 -$$

$$\frac{v}{2d} - \frac{v^2}{8d^2} \&c. = 2\sqrt{dv} \times 1 + \frac{v}{6d} + \frac{53v^2}{600d^2} \&c.$$

Now $2\sqrt{dv} \times 1 + \frac{v}{6d} + \frac{3v^2}{40d^2} \&c.$ is known to be the length of an arc whose diameter is d , and the versed sine of the half thereof v ; and this differs from $2\sqrt{dv} \times 1 + \frac{v}{6d} +$

$\frac{53v^2}{600d^2} \&c.$ in the third term only by $\frac{v^2}{75d^2}$, which shews the rule to be a near approximation. Q. E. I.

$\frac{2}{3} \text{ of}$

$$\frac{1}{3} \text{ of } CB = 12$$

$$\frac{4}{5} \text{ of } CB = 14.76$$

24

24

96

48

$$576 = \text{square of } DB$$

$$324 = \text{square of } CB$$

$$18)900(50 = \text{diameter } CF$$

90

0

50

14.76

$$35.24)12.000000(.3405$$

10572

14280

14096

18400

17620

780

1

.3405

1.3405

48

107240

53620

$$64.3440 = \text{arc } DCE.$$

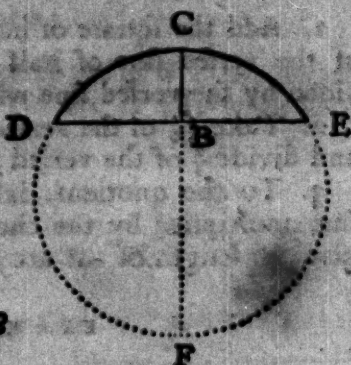
2. The chord of the whole arc is 7, and the versed sine 2: what is the length of the arc?

Ans. 8.439.

3. The chord of the whole arc is 40, and the versed sine 15: what is the length of the arc?

Ans. 53.62.

PRO-



PROBLEM XL

To find the area of a circle.

RULE I.*

Multiply half the circumference by half the diameter, and the product will be the area.

EXAMPLES.

1. What is the area of a circle whose diameter is 42, and circumference 131.946?

$$2)131.946$$

$$65.973 = \frac{1}{2} \text{ circumference.}$$

$$21 = \frac{1}{2} \text{ diameter.}$$

$$65973$$

$$131946$$

$$1385.433 = \text{area required.}$$

2. What is the area of a circle whose diameter is 10 feet, 6 inches, and circumference 31 feet, 6 inches?

* *Demon.* A circle may be considered as a regular polygon of an infinite number of sides, the circumference being equal to the perimeter, and the radius to the perpendicular. But the area of a regular polygon is equal to half the perimeter multiplied by the perpendicular, and consequently the area of a circle is equal to half the circumference multiplied by the radius, or half the diameter. Q. E. D.

15 *fe.*

$$\begin{array}{rcl} \text{fs.} & \text{in.} & \\ 15 & 9 = 15.75 = \frac{1}{2} \text{ circumference.} \\ 5 & 3 = 5.25 = \frac{1}{2} \text{ diameter.} \end{array}$$

7875

3150

7875

82.6875

12

8.2500

Ans. 82 feet, 8 inches.

3. What is the area of a circle whose diameter is 1, and circumference 3.14159? *Ans.* .7854

4. What is the area of a circle whose diameter is 7 and circumference 22? *Ans.* 38½

RULE II.

Multiply the square of the diameter by .7854, and the product will be the area.

EXAM-

Demon. All circles are to each other as the squares of their diameters. (Euc. XI. 2.)

But the area of a circle whose diameter is 1, is .7854 &c.
(by Rule I.) Therefore $1^2 : d^2 :: .7854 \text{ \&c.} : \frac{.7854 \text{ \&c.} \times d^2}{1^2}$

$= .7854 \text{ \&c.} \times d^2 = \text{area of a circle whose diameter is } d.$

Q. E. D.

The following proportions are those of *Metius* and *Archimedes*.

As 452 : 355 :: square of the diameter : Area.

As 14 : 11 :: square of the diameter : Area.

If the circumference be given, instead of the diameter, the area may be found as follows:

The

EXAMPLES.

1. What is the area of a circle whose diameter is 5?

$$\begin{array}{r} .7854 \\ 25 = \text{square of the diameter.} \end{array}$$

$$\begin{array}{r} 39270 \\ 15708 \\ \hline \end{array}$$

$$19.6350 = \text{the answer.}$$

2. What is the area of a circle whose diameter is 7? Ans. 38.4846

The square of the circumference $\times .07958 = \text{Area.}$

As 88 : 7 :: square of the circumference : Area.

As 1420 : 113 :: square of the circumference : Area.

And if d be the diameter, c the circumference, a the area, and $p = 3.14159$ &c. then:

$$1. d = \frac{c}{p} = \frac{4a}{c} = 2 \sqrt{\frac{a}{p}}$$

$$2. c = pd = \frac{4a}{d} = 2 \sqrt{pa}$$

$$3. a = \frac{pd^2}{4} = \frac{c^2}{4p} = \frac{dc}{4}$$

The following table will also shew most of the useful problems, relating to the circle and its equal or inscribed square.

1. diameter $\times .8862 = \text{side of an equal square.}$
2. circumf. $\times .2821 = \text{side of an equal square.}$
3. diameter $\times .7071 = \text{side of the inscribed square.}$
4. circumf. $\times .2251 = \text{side of the inscribed square.}$
5. area $\times .6366 = \text{side of the inscribed square.}$
6. side of a square $\times 1.4142 = \text{diameter of its circums. circle.}$
7. side of a square $\times 4.443 = \text{circumf. of its circums. circle.}$
8. side of a square $\times 1.128 = \text{diameter of an equal circle.}$
9. side of a square $\times 3.545 = \text{circumf. of an equal circle.}$

3. What

3. What is the area of a circle whose diameter is 4.5? *Ans.* 15.90435

4. How many square yards are there in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1.069016

PROBLEM XII.

To find the area of a sector, or that part of a circle which is bounded by any two radii and their included arc.

RULE.*

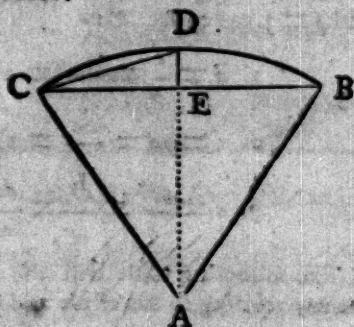
Multiply the radius, or half the diameter, by half the length of the arc of the sector, and the product will be the area.

Note. The length of the arc may be found by either of the last problems.

EXAMPLES.

1. The radius AB is 40, the chord BC of the whole arc 50, and the chord CD of half the arc 27; required the area of the sector.

$$\begin{array}{r}
 27 \\
 8 \\
 \hline
 216 \\
 50 \\
 \hline
 3)166 \\
 \hline
 2)55.3 \\
 \hline
 \end{array}$$



$$\begin{array}{l}
 27.6 = \frac{1}{2} \text{ the length of the arc } CDB. \\
 40 = \text{radius.}
 \end{array}$$

$$1106.6 = \text{area of the sector } ACDB.$$

* The rule for finding the area of the sector, is, evidently, the same as that for finding the area of the whole circle.

2. Required

2. Required the area of a sector of a circle, whose radius is 25, and the length of the arc 21.5.

Ans. 268.75.

3. What is the area of a sector whose radius is 10, the chord of the whole arc 20, and the chord of half the arc 11?

Ans. 113.3.

4. Find the area of the sector whose radius is 9, and the chord of whose arc is 6.

Ans. 27.5267835.

R U L E H.

As 360 is to the degrees in the arc of a sector, so is the area of the whole circle, whose radius is equal to that of the sector, to the area of the sector required.

Note. For a semi-circle, a quadrant, &c. take one half, one quarter, &c. of the whole area.

E X A M P L E S.

1. The radius of a sector of a circle is 20, and the degrees in its arc 22; what is the area of the sector?

* *Demon.* Let r = radius, d = number of degrees in the arc of the sector, and A = its area.

Then will $4r^2 \times .7854 = r^2 \times 3.1416$ = area of the whole circle, and $2r \times 3.1416$ = its circumference.

Also $360 : 2r \times 3.1416 :: d : \frac{2dr \times 3.1416}{360}$ = length of the arc of the sector. But $\frac{2dr \times 3.1416}{360} \times \frac{1}{2} \times r = \frac{dr^2 \times 3.1416}{360} = A$, by the last rule. And consequently $360 : d :: r^2 \times 3.1416 : A$.
Q. E. D.
 .7854

MENSURATION

.7854

 $1600 = \text{square of the diameter.}$

4712400

7854

 $360 : 22 :: 1256.6400 = \text{area of the whole circle.}$

22

251328

251328

 $36.0) 2764.608 (76.794 = \text{area of the sector.}$

252

244

216

286

252

340

324

168

144

24

2. Required the area of a sector whose radius is 25, and the length of its arc 147 degrees 29 minutes.

Ans. 804.4017.

3. Find the area of a semicircle whose radius is 13.

Ans. 265.4652.

4. Find the area of a quadrant whose radius is 21.

Ans. 346.3614.

P. R. O.

PROBLEM XIII.

To find the area of a segment of a circle,

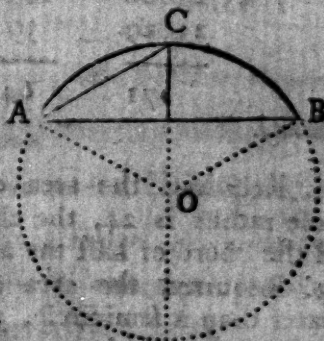
RULE I.

1. Find the area of the sector, having the same arc with the segment, by one of the last problems.
2. Find the area of the triangle formed by the chord of the segment, and the radii of the sector.
3. Then the sum, or difference, of these areas, according as the segment is greater or less than a semicircle, will be the area required.

EXAMPLES.

1. The radius OB is 10, the chord AB is 18, and the chord AC 10; what is the area of the segment ABC?

$$\begin{array}{r}
 10 = \text{chord } AC \\
 8 \\
 \hline
 80 \\
 18 = \text{chord } AB \\
 \hline
 3) 62 \\
 \hline
 2) 20.6 \\
 \hline
 10.3 = \frac{1}{2} \text{ arc } BCA \\
 10 = \text{radius} \\
 \hline
 103.3 = \text{area of the sector } OBCA
 \end{array}$$



$$\begin{array}{r} 18 \\ 10 \\ 10 \\ - \\ 2)38 \\ - \end{array}$$

$19 = \frac{1}{2}$ sum of the sides of $\triangle OBA$.

$$\begin{array}{r} 19 \cdot 19 \cdot 19 \\ 10 \cdot 10 \cdot 18 \\ \hline \end{array}$$

9 9 1 = three differences.

$19 \times 9 \times 9 = 1539 =$ product of $\frac{1}{2}$ sum and 3 differences.

$1539 (39.23 =$ area of the $\triangle OBA$.

9

$$\begin{array}{r} 69)639 \\ 621 \\ \hline \end{array}$$

$$\begin{array}{r} 782)1800 \\ 1564 \\ \hline \end{array}$$

$$\begin{array}{r} 7843)23600 \\ 23529 \\ \hline \end{array}$$

171

$103.33 =$ area of the sector.

$39.23 =$ area of the triangle.

$64.10 =$ area of the segment

required.

2. Required the area of a segment of a circle whose radius is 24, the chord of the whole arc 20, and the chord of half the arc 10.2. *Ans.* 28.07.

3. Required the area of a segment of a circle, greater than a semicircle, whose radius is 11.64, the chord of the whole arc 20.5, the chord of half the arc 20, and the chord of $\frac{1}{4}$ of the arc 11.5.

Ans. 335.89.

4. What

4. What is the area of a segment, whose arc is a quadrant, the diameter being 18 feet?

Ans. 23.1174.

5. What is the area of a segment of a circle whose arc contains 280 degrees, the diameter being 50?

Ans. 1834.9191.

RULE II.

1. Add the square of half the chord of the segment to the square of its height, and multiply the square root of the sum by 4.

2. To

* *Demon.* Let $c = BD$, $v = CD$, and $d = \text{diameter } CE$; then will the rule be expressed by $2c + \frac{4}{3}\sqrt{c^2 + v^2} \times \frac{4}{15}v$. But $2c + \frac{4}{3}\sqrt{c^2 + v^2} \times \frac{4}{15}v = \sqrt{dv - v^2} + \frac{4}{3}\sqrt{dv} \times \frac{4}{15}v = \frac{4}{3}v \times \frac{4}{3}\sqrt{dv} + \frac{4}{3}v \sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3} \&c. = \frac{4}{3}v \sqrt{dv} - \frac{2v^2 \sqrt{dv}}{5d} - \frac{2v^3 \sqrt{dv}}{20d^2} \&c. = 2v \sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{20d^2} \&c.$

Again, since $BD = \sqrt{dv - v^2}$, and $DC = v$, therefore $v \sqrt{dv} \times 1 - \frac{v}{2d} - \frac{v^2}{8d^2} - \frac{3v^3}{48d^3} \&c.$ will be the fluxion of the segment; and its fluent, $v \sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} \&c.$ will be the value of half the segment. Consequently its double, $2v \sqrt{dv} \times \frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} \&c.$ will be the value of the whole segment; which differs from the expression for the rule only in the last term, and shews the approximation required. Q. E. D.

This rule was first given by *Sir Isaac Newton*, and published by *Jones, Wallis, and Colson*.

Some of the following methods are more accurate, but none of them so convenient in practice.

2. To $\frac{1}{3}$ of the number last found, add the whole chord of the segment, and this sum multiplied by $\frac{2}{3}$ of the height will give the area required.

EXAMPLES.

1. The chord AB of the segment BCA is 40, and its height, or versed sine, DC is 10: what is the area of the segment?

20

20

400 = square of $\frac{1}{3}$ the chord AB.

100 = square of the height DC.

500 (22.36

4

42) 100

84

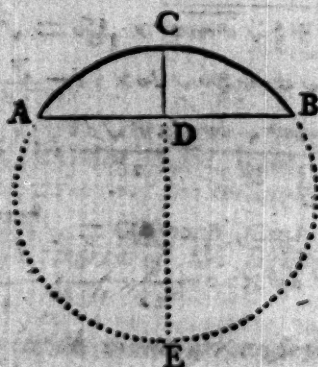
443) 1600

1329

4466) 27100

26796

304



I. $\frac{2}{3}v\sqrt{c^2 + \frac{3}{4}v^2}$ = area nearly.

II. $2\sqrt{4c^2 + v^2} + \sqrt{c^2 + v^2} \times \frac{4}{15}v$ = area very nearly.

III. $7\sqrt{c^2 + \frac{3}{4}v^2} + \frac{4}{3}\sqrt{c^2 + v^2} \times \frac{4}{15}v$ = area extremely near.

22.36

$$\begin{array}{r}
 22.36 \\
 4 \\
 \hline
 3)89.44 \\
 \hline
 29.81 \\
 40 \\
 \hline
 69.81 \\
 4 = \frac{2}{3} \text{ of the height.} \\
 \hline
 279.24 = \text{area required.}
 \end{array}$$

2. The chord is 20, and the height, or versed sine 2; required the area of the segment.

Ans. 26.877908.

3. The length of the chord is 48, and the height of the segment 18; what is the area? *Ans.* 633.6.

R U L E III. *

1. Divide the height, or versed sine, by the diameter, and find the quotient in the table of versed sines.

2. Mul-

* The table to which this rule refers, is formed of the areas of the segments of a circle whose diameter is 1; and which is supposed to be divided by perpendicular chords into 1000 equal parts.

The reason of the rule itself depends upon this property.--- That the versed sines of similar segments are as the diameters of the circles to which they belong, and the areas of those segments as the squares of the diameters; which may be thus proved.

Let $ADBA$ and $adba$ be any two similar segments, cut off from the similar sectors $ADBCA$ and $adbCa$, by the chords AB and ab ; and let fall the perpendicular CD .

E 3

Then,

2. Multiply the number on the right hand of the verfed sine, by the square of the diameter, and the product will be the area.

EXAMPLES.

1. If the chord of a circular segment be 40, its verfed sine 10, and the diameter of the circle 50: what is the area?

$$5.0 \overline{) 1.0}$$

.2 = tabular verfed sine.

.111823 = tabular segment.

2500 = square of 50.

$$\begin{array}{r} 55911500 \\ 223646 \\ \hline \end{array}$$

279.557500 = area required.

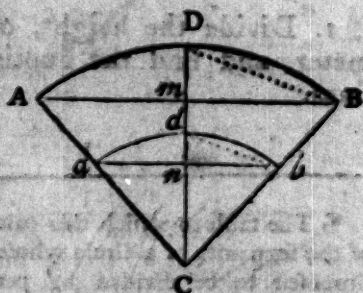
Then, by similar triangles, $DB:db::BC:bC$, and $DB:db::Dm:dn$; whence, by equality, $BC:bC::Dm:dn$, or $2BC:2bC::Dm:dn$.

Again, since similar segments are as the squares of their chords, it will be $AB^2:a b^2::ADBA:adba$; but $AB^2:a b^2::CB^2:C b^2$, and therefore, by equality, $ADBA:adba::CB^2:C b^2$, or $ADBA:adba::4CB^2:4C b^2$. Q. E. D.

Now, if d be put equal to any diameter, and v the verfed sine, it will be $d:v::1$ (diameter in the table): $\frac{v}{d}$ = verfed sine of a similar segment in the table, whose area let be called a .

Then $1^2:d^2::a:a d^2$ = area of the segment whose height is v , and diameter d , as in the rule.

2. The



2. The chord of the segment is 20, the versed sine 5, and the diameter 25: what is the area?

Ans. 69.889375.

PROBLEM XIV.

To find the area of a circular zone, the breadth of the zone, and the length of its two parallel sides being known.

RULE.

1. Add the square of half the difference of the two parallel sides to the square of the zone's breadth, and the square root of the sum will be equal to the chord of either of the circular parts of the zone.

2. Mul-

* A circular zone is the space included between any two parallel chords and their intercepted arcs.

Demon. of the rule. Let ABCDA be any zone whatever, and *o* the center of the circle; draw the chords DA and CB, and let fall the perpendiculars *Drm* and *one*; join the points *mB* and *mC*, and draw the diameter *aso*.

Now, since the angle *CDm* is a right angle, *mC* will also be the diameter of the circle, and *on* will be parallel to and equal to half *mB*. Put *Br* ($= \frac{1}{2}$ sum of AB, DC) = *S*, and *Ar* ($= \frac{1}{2}$ difference AB, DC) = *D*; then will $\sqrt{Dr^2 + rA^2} = \sqrt{as^2 + D^2} = AD$ or *BC*. But the triangles *Dra* and *mrb* are similar, and therefore *Dr* (*as*) : *AD* ($\sqrt{as^2 + D^2}$) :: *Br* (*S*) : *Bm* = $S \times \frac{as}{\sqrt{as^2 + D^2}}$; and

consequently $S \times \frac{as}{\sqrt{as^2 + D^2}} = on$. Whence (Euc. I. 47) $\frac{1}{2}$

$\sqrt{D^2 + as^2 + S^2 \times \frac{D^2 + as^2}{as^2}} = \text{radius } oc \text{ or } oe$; and $\frac{1}{2}$

$\sqrt{D^2 + as^2 + S^2 \times \frac{D^2 + as^2}{as^2}} - S^2 \times \frac{D^2 + as^2}{2as^2} = ne$

E 4

height

2. Multiply the square of this chord by the square of half the sum of the sides, and divide the product by the square of the breadth of the zone, and to this quotient add the square of the said chord, and half the square root of the sum will give the radius of the circle.

3. Multiply half the sum of the sides by the chord last found, and divide the product by twice the breadth of the zone; and this quotient taken from the radius, will give the height of the segment which stands upon the said chord.

4. Find the area of that part of the zone which forms a trapezoid, by Prob. 6. and the area of the segment by Prob. 13. Rule 3.

5. Add the area of the trapezoid to twice the area of the segment, and it will give the area of the zone required.

EXAMPLES.

1. The greater chord or side AB is 96, the less side DC 60, and the breadth a is 26: what is the area of the zone ABCDA?

height of the segment; which is the same as in the rule. And, since the two segments are equal, it is evident that the double of either of them being added to the trapezoid ABCD will give the area of the zone. Q. E. D.

Coroll. I. When the two parallel sides of the zone are equal, the chord of the segment will be equal to the breadth of the zone, and the height of the segment will be $\sqrt{\frac{1}{4}AB^2 + \frac{1}{4}a^2} - \frac{1}{2}AB$.

Coroll. II. When one of the sides is the diameter of the circle, the chord will be $\sqrt{a^2 + D^2}$, and the height of the segment $\frac{1}{2}AB - \sqrt{\frac{1}{4}AB^2 - \frac{1}{4}a^2} - \frac{1}{2}D^2$.

1. To find the radius of the circle.

$\frac{1}{2}$ the sum of the sides is = 78.

$\frac{1}{2}$ the diff. of the sides is = 18.

18

18

—

144

18

—

324 = square of $\frac{1}{2}$ diff. of sides.

676 = square of the zone's breadth.

1000 (31.6227 = chord AD or BC.

9

61)100

61

—

626)3900

3756

—

6322)14400

12644

—

63242)175600

126484

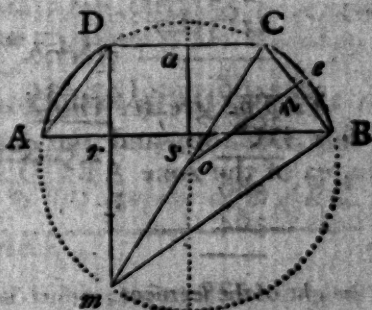
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632447)4811600

4427129

—

384471



MENSURATION

1000 = square of the chord AD.

6084 = square of $\frac{1}{2}$ sum of the sides.

$$\begin{array}{r} 676 \overline{) 6084000} (9000 \\ \underline{6084} \quad \underline{1000} \quad (2) \\ 10000 \quad (100 \end{array}$$

50 = radius oC or oe.

2. To find the height of the segment C B n.

31.6227 = chord BC.

78

2529816

2213589

50.000

52) 2466.5706 (47.434

208

2.566 = n e = height of the segment

386

364

225

208

177

156

210

208

26

3. To

3. To find the area of the segment C B n.

$$100)2.566$$

$$.02566 = \text{tabular versed sine.}$$

$$.005230 = \text{tabular segment for } .025.$$

$$.005546 = \text{ditto for } .026.$$

$$.000316$$

$$.66$$

$$1896$$

$$1896$$

$$.00020856$$

$$.005230$$

$$.00543856 = \text{ditto for } .025^{\frac{66}{100}}, \text{ or } .02566.$$

$$10000 = \text{square of the diameter.}$$

$$54.38560000 = \text{area of the required segment C B n.}$$

4. To find the area of the trapezoid D A B C.

$$156 = \text{sum of the sides.}$$

$$26 = \text{breadth.}$$

$$936$$

$$312$$

$$2)4056$$

$$2028 = \text{area of the trapezoid.}$$

$$108.7712 = \text{twice area of the segment C B n.}$$

$$2136.7712 = \text{area of the zone, or answer required.}$$

2. The greater side is 20, the lesser 15, and the breadth $17\frac{1}{2}$: what is the area of the zone?

Ans. 395.4369.

3. One end of a circular zone is 48, the other end is 30, and the breadth is 13: what is the area of the zone?

Ans. 534.4249.

PROBLEM XV.

To find the area of a circular ring, or the space included between the circumferences of two concentric circles.

RULE.*

Multiply the sum of the diameters by their difference, and this product again by .7854, and it will give the area required.

EXAMPLES.

1. The diameters AB and CD are 20 and 15: required the area of the circular ring, or the space included between the circumferences of those circles.

* *Demon.* The area of the circle AIBA = $AB^2 \times .7854$, and the area of the small circle CD is = $CD^2 \times .7854$; therefore the area of the ring = $AB^2 \times .7854 - CD^2 \times .7854 = AB + CD \times AB - CD \times .7854$. Q. E. D.

Coroll. If CI be a perpendicular at the point C, then will the area of the ring be equal to that of a circle whose radius is CI.

Rule 2. Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.

20

15

—

35 = *sum of the diameters.*5 = *difference of ditto.*

175

.7854

—

700

875

1400

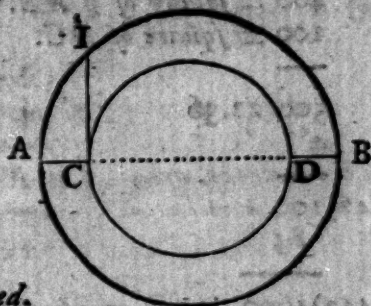
1225

137.4450 = *area required.*

2. The diameters of two concentric circles are 16 and 10 : what is the area of the ring formed by those circles?

Ans. 122.5224.

3. The two diameters are 21.75 and 9.5 : required the area of the circular ring.

Ans. 300.6609.

PROBLEM XVI.

To find the areas of lunes, or the spaces included between the intersecting arcs of two eccentric circles.

RULE.*

Find the areas of the two segments from which the lune is formed, and their difference will be the area required.

EXAMPLES.

* Whoever wishes to be acquainted with the properties of lunes, and the various theorems arising from them, may consult *Mr. Whiston's Commentary on Tacquet's Euclid*, where they will find this subject very ingeniously managed.

The

EXAMPLES:

The length of the chord AB is 40, the height DC 10, and DE 4: required the area of the lune AC BEA.

$$400 = \text{square of } \frac{1}{2} AB.$$

$$100 = \text{square of } DC.$$

$$500(22.36$$

$$4$$

$$42)100$$

$$84$$

$$443)1600$$

$$1329$$

$$4466)27100$$

$$26796$$

$$304$$

$$22.36$$

$$4$$

$$3)89.44$$

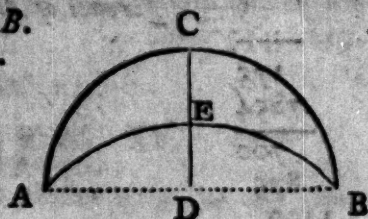
$$29.81$$

$$40$$

$$69.81$$

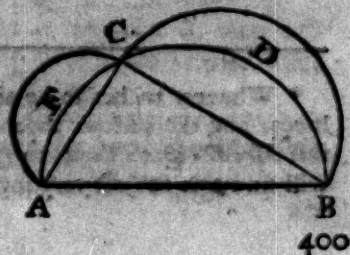
$$4 = \frac{2}{3} \text{ of the height.}$$

$$279.24 = \text{area of the seg. } ABCA.$$



The following property is one of the most curious.

If ABC be a right angled triangle, and semicircles be described on the three sides as diameters, then will the said triangle be equal to the two lunes D and F taken together.



$$400 = \text{square of } \frac{1}{2} AB$$

$$16 = \text{square of } DE$$

$$416(20.396$$

$$4$$

$$403)1600$$

$$1209$$

$$20.396$$

$$4$$

$$4069)39100$$

$$36621$$

$$3)81.584$$

$$40786)247900$$

$$244716$$

$$27.194$$

$$40.$$

$$3184$$

$$67.194$$

$$1.6 = \frac{2}{3} \text{ of the height.}$$

$$403164$$

$$67194$$

$$107.5104 = \text{area of the seg. } ABEA.$$

$$279.24$$

$$107.5104$$

$$171.7296 = \text{area of the lune required.}$$

2. The chord is 20, and the heights of the segments 10 and 2: required the area of the lune.

Ans. 128.522

3. The length of the chord is 48, and the heights of the segments 18 and 7: what is the area of the lune?

Ans. 405.8676.

PROBLEM XVII.

To find the area of an irregular polygon, or a figure of any number of sides.

RULE.

R U L E . *

1. Divide the figure into triangles and trapeziums, and find the area of each separately.
2. Add these areas together, and the sum will be equal to the area of the whole polygon.

E X A M P L E S .

1. Required the area of the irregular figure ABCDEFGA, the following lines being given.

$$\begin{array}{lll} \text{GB} = 30.5, & \text{As} = 11.2, & \text{Cs} = 6 \\ \text{GD} = 29 & \text{Fs} = 11 & \text{Cs} = 6.6 \\ \text{FD} = 24.1 & \text{Es} = 4 & \end{array}$$

$$\begin{array}{r} 11.2 = \text{As} \\ 6 = \text{Cs} \\ \hline \end{array}$$

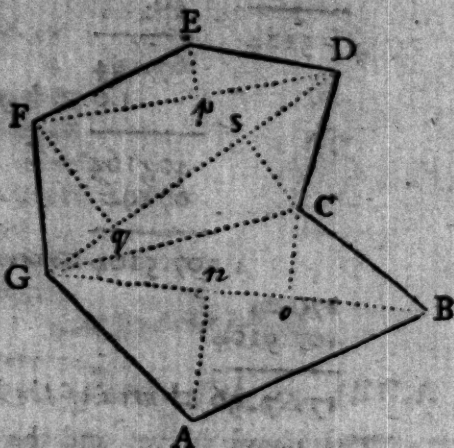
$$2) 17.2$$

$$\hline 8.6$$

$$30.5 = \text{GB}$$

$$\begin{array}{r} 430 \\ 2580 \\ \hline \end{array}$$

$$262.30 = \text{area of } ABCG.$$



* When any part of the figure is bounded by a curve, the area may be found as follows :

Rule. 1. Erect any number of perpendiculars upon the base, at equal distances, and find their lengths.

2. Add the lengths of the perpendiculars, thus found, together, and the sum divided by their number will give the *mean breadth*.

3. Multiply the mean breadth by the length of the base, and it will give the area of that part of the figure required.

$$11 = F,$$

$$6.6 = C,$$

$$2) 17.6$$

$$8.8$$

$$29 = GD$$

$$792$$

$$176$$

$$255.2 = \text{area of } GCDP$$

$$24.8 = FD$$

$$4 = EP$$

$$2) 99.2$$

$$49.6 = \text{area of } FDE$$

$$262.30$$

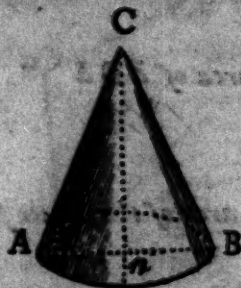
$$255.2$$

$$567.10 = \text{area of the whole figure required.}$$

OF THE CONIC SECTIONS.

DEFINITIONS.

1. **T**HE *conic sections* are such plain figures as are formed by the cutting of a cone.
2. * A *cone* is a solid described by the revolution of a right-angled triangle about one of its legs, which remains fixed.



3. The *axis of a cone* is the right line about which the triangle revolves.

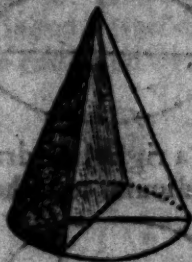
* This is Euclid's definition of a cone, and is that which it is apprehended will be best understood by a learner; but the following one is more general.

Conceive the right line CB to move upon the fixed point C as a center, and so as continually to touch the circumference of the circle AB, placed in any position, except in that of a plane which passes through the said point; and then that part of the line which is intercepted between the fixed point and the periphery of the circle will generate the convex superficies of a cone.

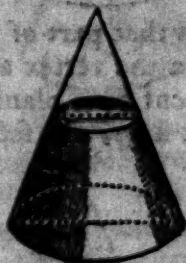
4. The

4. The *base* of a cone is the circle which is described by the revolving leg of the triangle.

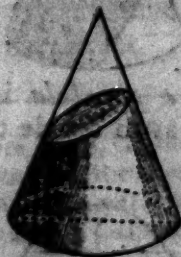
5. If a cone be cut through the vertex, by a plane perpendicular to that of the base, the section will be a *triangle*.



6. If a cone be cut into two parts, by a plane parallel to the base, the section will be a *circle*.



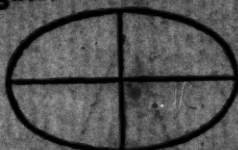
7. If a cone be cut by a plane which passes through its two slant sides in an oblique direction, the section will be an *ellipsis*.



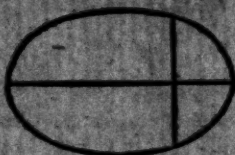
8. If two lines be drawn through the center of an ellipsis, perpendicular to each other, and terminated both

both ways by the circumference, they are called the *transverse* and *conjugate diameters*.

The longest line is the transverse axis, and the shortest the conjugate.

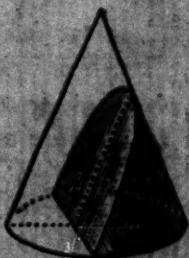


9. An *ordinate* is a right line drawn from any point of the curve, perpendicular to either of the diameters.

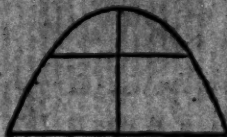


10. An *abscissa* is that part of the diameter which is contained between the vertex and the ordinate.

11. If a cone be cut by a plane, which is parallel to either of its slant sides, the section will be a *parabola*.



12. The *axis* of a *parabola* is a right line drawn from the vertex, so as to divide the figure into two equal parts.

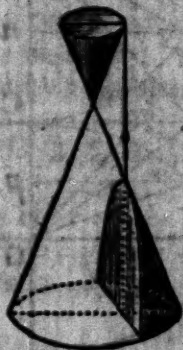


13. The

13. *The ordinate* is a right line drawn from any point in the curve perpendicular to the axis.

14. *The abscissa* is that part of the axis which is contained between the vertex and the ordinate.

15.* If a cone be cut into two parts, by a plane, which being continued, would meet the opposite cone, the section is called an *hyperbola*.



16. *The transverse diameter* of an hyperbola is that part of the axis intercepted between the two opposite cones.

17. *The semi-conjugate diameter* is a perpendicular drawn from the vertex of the cone to the transverse axis.

18. *An ordinate* is a line drawn from any point in the curve perpendicular to the transverse diameter; and the *abscissa* is the distance intercepted between that ordinate and the vertex.

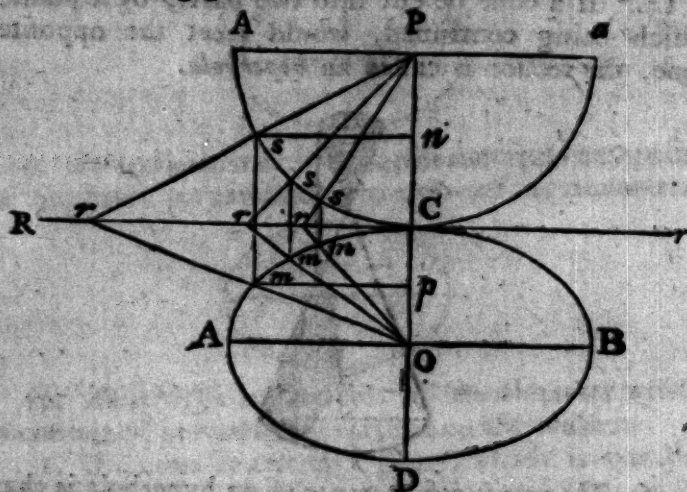
* The two opposite cones, in this definition, are supposed to be generated together, by the revolution of the same line.

All the figures which can possibly be formed by the cutting of a cone, are mentioned in these definitions, and are the five following ones; viz. a *triangle*, a *circle*, an *ellipsis*, a *parabola*, and an *hyperbola*; but the three last only are usually called the *conic sections*.

P R O.

PROBLEM I.

To describe an ellipsis, the transverse and conjugate diameters being given.



Construction. 1. Make AB and CD equal to the transverse and conjugate diameters, and draw the indefinite right line RCr parallel to AB.

2. In

Demon. Let the transverse diameter AB = t , the conjugate CD = c , $cp = x$, and $mp = sn = y$.

Then since $Pn = \left[\frac{t^2}{2} - y^2 \right]^{\frac{1}{2}} = \frac{\sqrt{t^2 - 4y^2}}{2}$, and the triangles

Pn and PCr are similar, we shall have $\frac{\sqrt{t^2 - 4y^2}}{2} : y :: \frac{t}{2} : \frac{cy}{\sqrt{t^2 - 4y^2}} = Cr$.

Again, as the triangles Opm , and OCr are also similar, it will be $\frac{c}{2} - x : y :: \frac{c}{2} : \frac{cy}{c - 2x} = Cr$.

And

2. In DC produced take CP equal to the semi-transverse AO, and, with the radius PC, describe the semi-circle ACA.

3. From the point P draw any number of lines Psr , Psr , &c. cutting Rr in the points r , r , r , &c. and the semi-circle in s , s , s , &c.

4. Join Or , Or , Or , &c. and draw the lines sm , sm , sm , &c. parallel to the conjugate diameter CD, and they will cut the corresponding lines Or , Or , Or , &c. in the points m , m , m , &c. which are in the circumference of the ellipsis required.

PROBLEM II.

In an ellipsis, any three of the four following terms being given, viz. the transverse and conjugate diameters, an ordinate and its abscissa, to find the fourth.

CASE I.

When the transverse, conjugate, and abscissa are known, to find the ordinate.

RULE.

From the square of the half transverse, take the square of the abscissa, and multiply the square root

And consequently $\frac{cy}{c-2x} = \frac{ty}{\sqrt{t^2-4y^2}}$, and from hence

$y^2 = \frac{c^2}{t^2} \times tx - x^2$, which is the known equation for an ellipse. Q. E. D.

* Let t = transverse diameter, c = conjugate, x = abscissa, and y = ordinate as before. Then will the general equation expressing the property of the ellipse be $t^2 : c^2 :: x \times t - x^2 : y^2$; and from this the four rules here given are deduced.

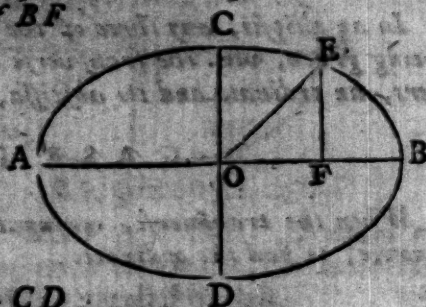
of

of the remainder, by the quotient of the half conjugate divided by the half transverse, and the product is the ordinate.

EXAMPLES.

1. In the ellipsis ADBC, the transverse diameter AB is 120, the conjugate diameter CD is 40, and the abscissa BF 36: what is the length of the ordinate EF?

$$\begin{array}{r}
 60 \\
 60 \\
 \hline
 3600 = \text{square of } \frac{1}{2} AB \\
 1296 = \text{square of } BF \\
 \hline
 2304(48 \\
 16 \\
 \hline
 88)704 \\
 704
 \end{array}$$



$$\frac{1}{2} AB = 60)20 = \frac{1}{2} CD$$

.3333, &c. or $\frac{1}{3}$, and $\frac{1}{3}$ of 48 = 16 = ordinate EF, required.

2. If the transverse diameter be 35, the conjugate 25, and the abscissa 28: what is the ordinate?

Ans. 10.

CASE II.

When the transverse, conjugate, and ordinate are known, to find the abscissa.

RULE.

From the square of the half conjugate, take the square of the ordinate, and multiply the square root of

of the remainder by the quotient of the half transverse divided by the half conjugate, and the product is the abscissa.

EXAMPLES.

1. The transverse diameter AB is 120, the conjugate diameter CD is 40, and the ordinate FE is 16: what is the abscissa?

20

20

400 = square of $\frac{1}{2}$ CD

256 = square of EF

144 (12

144

$\frac{1}{2}$ CD = 20) 60 = $\frac{1}{2}$ AB

3

12

36 = abscissa FB required.

2. What are the two abscissas to the ordinate 20, the diameters being 35 and 25?

Ans. 7 and 28.

CASE III.

When the conjugate, ordinate, and abscissa are known, to find the transverse.

RULE.

From the square of the half conjugate take the square of the ordinate, and divide the product of the half

F

half

half conjugate and abscissa by the square root of the remainder, and the quotient will be the half transverse.

EXAMPLES.

1. The conjugate diameter CD is 40, the ordinate EF is 16, and the abscissa FB 36: required the transverse AB .

$$20$$

$$20$$

$$400 = \text{square of } \frac{1}{2} CD.$$

$$256 = \text{square of } EF.$$

$$144$$

$$144$$

$$36 = FB.$$

$$20 = \frac{1}{2} CD.$$

$$12 \overline{) 720}$$

$$60 = \frac{1}{2} AB, \text{ and } 120 = \text{transverse diameter } AB \text{ required.}$$

2. If an ordinate and its lesser abscissa be 10 and 7, and the conjugate 25: what is the transverse?

Ans. 35.

CASE IV.

The transverse, the ordinate, and the abscissa being given, to find the conjugate.

RULE.

From the square of the half transverse take the square of the abscissa, and divide the product of the half

half transverse and ordinate by the square root of the remainder, and the quotient will give the half conjugate.

EXAMPLES.

1. The transverse AB is 120, the ordinate EF 16, and the abscissa FB 36: required the conjugate.

$$\begin{array}{r} 60 \\ 60 \\ \hline \end{array}$$

$$3600 = \text{square of } \frac{1}{2} AB.$$

$$1296 = \text{square of } FB.$$

$$\begin{array}{r} 2304 \\ 16 \\ \hline \end{array}$$

$$\begin{array}{r} 88)704 \\ 704 \\ \hline \end{array}$$

$$16 = EF.$$

$$60 = \frac{1}{2} AB.$$

$$48)960 (20 = \frac{1}{2} CD, \text{ and } 40 = \text{conjugate } CD \text{ required.}$$

•

2. The transverse diameter is 35, the ordinate is 10, and its abscissa 7: what is the conjugate?

Ans. 25.

PROBLEM III.

To find the circumference of an ellipse.

F 2

RULE.

R U L E . *

Multiply the square root of half the sum of the squares of the two diameters by 3.1416, and the product will be the circumference *nearly*.

* *Demon.* Let t = transverse diameter, c = conjugate, $p = 3.1416$, and $d = 1 - \frac{c^2}{t^2}$. Then will $pt \times : 1 - \frac{d}{2.2} - \frac{d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8}$ &c. = circumference of the ellipse, as is shewn by the writers on fluxions.

Now the rule given above is $p\sqrt{\frac{t^2+c^2}{2}} = p \times \sqrt{\frac{1}{2} + \frac{c^2}{2t^2}}$
 $\times t^2 = pt \sqrt{\frac{1}{2} + \frac{c^2}{2t^2}} = pt \sqrt{1 - \frac{1}{2} + \frac{c^2}{2t^2}} =$ (by substitution)
 $pt \sqrt{1 - \frac{d}{2}} = pt \times : 1 - \frac{d}{2.2} - \frac{d^2}{2.2.4} - \frac{3d^3}{2.2.4.6}$ &c. But the three first terms of this series differ from the three first terms of the former only by $\frac{d^2}{64}$; and therefore the rule is shewn to be an approximation. Q. E. D.

Rule 2. Multiply $\frac{1}{2}$ the sum of the two diameters by 3.1416, and the product will give the circumference *exact enough to answer most practical purposes*.

Rule 3. Find the circumference both from the last rule and that given above, and $\frac{1}{2}$ the sum of the results will give the circumference *extremely near*.

Note. If a = semi-transverse B O, c = semi-conjugate C O, and x = distance O F, of the ordinate E F from the center, then will the arc C E be $= x \times : 1 + \frac{c^2}{6a^4} x^2 + \frac{4a^2c^2 - c^4}{40a^8} x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}} x^6$ &c.

The following may serve as a practical rule for finding the length of the arc C E.

Find the length of a circular arc intercepted by O E and O C, and whose radius is $\frac{1}{2}$ the sum of O E and O C, and it will be the elliptic arc C E *nearly*.

E X A M .

CONIC SECTIONS.

EXAMPLES.

1. The tranverse diameter is 24 and the conjugate 20: required the circumference of the ellipse.

24

24

—

96

48

—

576 = square of the tranverse.

400 = square of the conjugate.

—

2)976

—

488 (22.09

4

—

43)88

84

—

4409)40000

39681

—

319

3.1416

22.09

—

382744

628320

62832

69.397944 = circumference required.

2. The axes are 24 and 18: what is the circumference?

Ans. 66.6433.

CONIC SECTIONS.

PROBLEM IV.

To find the area of an ellipse.

RULE.

Multiply the transverse diameter by the conjugate, and the product again by .7854, and it will give the area required.

EXAMPLES.

1. What is the area of an ellipse whose transverse diameter is 24, and the conjugate 18?

24 = transverse.

18 = conjugate.

$$\begin{array}{r} 24 \\ \times 18 \\ \hline 192 \\ 432 \\ \hline 432 \\ \times .7854 \\ \hline 1728 \\ 2160 \\ \hline 3456 \\ 3024 \\ \hline \end{array}$$

339.2928 = area required.

2. If the axes of an ellipse be 35 and 25, what is the area?

Ans. 687.225.

The demonstration of this rule is contained in that of the next problem.

PROBLEM V.

To find the area of an elliptic segment, whose base is parallel to either of the axes of the ellipse.

RULE.

Divide the height of the segment by that axe of the ellipse of which it is a part, and find, in the table, a circular segment whose versed sine shall be equal to this quotient; then the segment thus found, and the two axes of the ellipse being multiplied continually together, will give the area required.

EXAMPLES.

1. Required the area of the elliptic segment EAF, whose height AG is 10, and the axes of the ellipse AB and CD, 35 and 25 respectively.

Demon. Let the transverse diameter $AB = a$, the conjugate $CD = c$, $AG = x$, and $EG = y$; then by the property of the curve we shall have $y = \frac{c}{a} \sqrt{ax - x^2}$, and the fluxion

of the area EAF $= (y\dot{x}) = \frac{c}{a} \times \dot{x} \sqrt{ax - x^2}$. But $\dot{x}$

$\sqrt{ax - x^2}$ is known to express the fluxion of the corresponding circular segment, whose versed sine is x , and diameter a . Let the fluent of this expression therefore be denoted by A ,

and then the fluent of $\frac{c}{a} \times \dot{x} \sqrt{ax - x^2}$ will be $= \frac{c}{a} \times$

A , from whence the rule is formed. Q. E. I.

Coroll. The ellipse is equal to a circle whose diameter is a mean proportional between the two axes, and from hence the rule is formed for the whole ellipse.

35)10,0000
 2857 = *tabular versed sine*, and for the vertex
 And the segment belonging to this is 184521

35 = AB.

922605
 753563

8.458235
 25 = CD.

42291175
 16916470

211.455875 = *area required*.

2. What is the area of an elliptic segment cut off by a double ordinate parallel to the conjugate axis, at the distance of 36 from the center, the axes being 120 and 40?

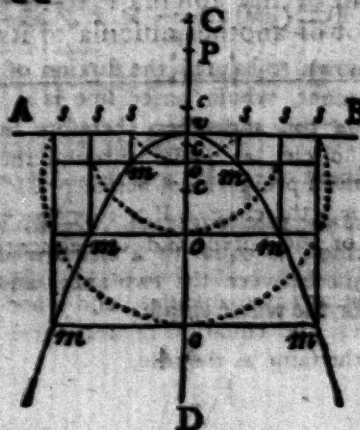
Ans. 536.7504.

3. What is the area of a segment, cut off by an ordinate parallel to the transverse diameter, whose height is 5, the axes being 35 and 25?

Ans. 97.845125.

PROBLEM VI.

To describe a parabola, any ordinate to the axis and its abscissa being given.



CON.

* CONSTRUCTION. 1. Draw two indefinite right lines AB and CD perpendicular to each other, and let the point of their intersection v be assumed for the vertex.

2. From v set off vP = to the parameter, or a third proportional to the given abscissa and its ordinate.

3. Take any number of points c, c, c , &c. in the line CD, and on them with the distances cP, cP, cP , &c. describe the circles sos, sos, sos , &c. cutting AB in the points s, s, s , &c. and CD in o, o, o , &c.

4. Draw mom, mom, mom , &c. parallel to AB, and sm, sm, sm , &c. parallel to CD, and the points m, m, m , &c. of their intersection will be so many points in the curve of the parabola required.

PROBLEM VII.

In a parabola, any three of the four following terms being given, viz. any two ordinates and their two abscissas, to find the fourth.

RULE. †

As any abscissa is to the square of its ordinate, so is any other abscissa to the square of its ordinate; or as the square root of any abscissa is to its ordinate, so is the square root of another abscissa to its ordinate.

* Let $vP = p$, $vo = x$, and $om = y$.

Then since the circle sos passes through the point P we shall have vs a mean proportional between vP and vo ; but vs is equal to om , therefore $vP \times vo = om^2$, or $px = y^2$ for the equation of the curve, and consequently mom is a parabola.

† If x and X be any two abscissas, and y and Y their corresponding ordinates, the equation of the curve will be $xY^2 = XY^2$, which is the same as the rule.

EXAMPLES.

1. The abscissa VF is 9, and its ordinate EF 6; required the ordinate GH, whose abscissa VH is 16.

The square root of 9 is 3, and that of 16, 4.

Therefore $3:6::4$

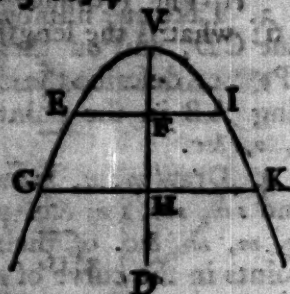
6

—
3)24

8 = ordinate

GH.

Or



As $9 (VF) : 36 (\text{square of } EF) :: 16 (VH)$

36

—
96

48

—
9)576

64 = square of

GH, whose root is 8, the same as before.

2. The two abscissas are 9 and 16, and their corresponding ordinates 6 and 8; from any three of these to find the fourth.

PROBLEM VIII.

To find the length of any arc of a parabola, cut off by a double ordinate.

RULE.

CONIC SECTIONS.

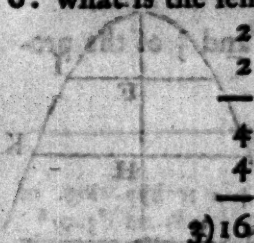
107

RULE.*

To the square of the ordinate add $\frac{1}{4}$ of the square of the abscissa, and twice the square root of the sum will be the length of the curve required.

EXAMPLES.

1. The ordinate VH is 2, and its ordinate GH 6: what is the length of the arc GVK ?



$$\begin{array}{r} 2 \\ 2 \\ \hline \end{array}$$

$$4 = \text{square of } VH$$

$$\begin{array}{r} 4 \\ \hline \end{array}$$

$$3)16$$

$$\text{Square of } GH = 36 \quad 5.333 = \frac{1}{4} \text{ square of } VH$$

$$\begin{array}{r} 41.333333 \\ 36 \end{array} \quad \begin{array}{r} 6.429 \\ 2 \end{array}$$

$$\begin{array}{r} 124)533 \\ 496 \end{array} \quad 12.858 = \text{length of the arc nearly.}$$

$$\begin{array}{r} 1282)3733 \\ 2564 \end{array}$$

$$\begin{array}{r} 12849)116933 \\ 115641 \end{array}$$

$$1292$$

PROBLEM

* Demon. Let x = any abscissa, y = its ordinate, $a = \frac{1}{2}$ the parameter of the axe, and $g = \frac{y}{a}$. Then it is shewn by the

F 6.

writers

PROBLEM IX.

To find the area of a parabola.

RULE.

Multiply the base by the height, and $\frac{2}{3}$ of the product will be the area required.

writers on fluxions, that $xy \sqrt{1+y^2} + x \times \text{hyp. log. of}$
 $y + \sqrt{1+y^2} = 2y \times 1 + \frac{y^2}{2.3} - \frac{y^4}{2.4.5} + \frac{3y^6}{2.4.6.7} - \frac{3.5y^8}{2.4.6.8.9}$

&c. = c = length of the curve. But $\sqrt{1+\frac{1}{3}y^2} = 1 + \frac{y^2}{2.3}$
 $-\frac{y^4}{2.4.9} + \frac{3y^6}{2.4.6.27}$ &c.

Therefore $\frac{c}{2y} = \sqrt{1+\frac{1}{3}y^2}$ nearly. And consequently $c =$

$2y \sqrt{1+\frac{1}{3}y^2} = \sqrt{y^2 + \frac{4}{3}y^3}$ the same as the rule. Q. E. D.

The following rule is a still nearer approximation:

$9\sqrt{y^2 + \frac{4}{3}y^3} - 4 \times \frac{y^2 + \frac{2}{3}y^3}{x} \times \frac{2}{3} = \text{length of the arc ex-}$
trremely near,

* *Demon.* Let $VH = x$, $GH = y$, and the parameter = p .

Then $px = y^2$ or $px^{\frac{1}{2}} = y$ by the nature of the curve.

Whence the fluxion of the area ($= yx$) $= px^{\frac{1}{2}} x$, and its flu-

ent $= \frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3} p^{\frac{1}{2}} x^{\frac{1}{2}} \times x$.

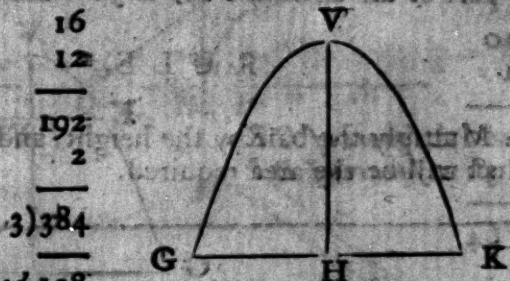
But because $y = p^{\frac{1}{2}} x^{\frac{1}{2}}$, therefore $\frac{2}{3} p^{\frac{1}{2}} x^{\frac{1}{2}} \times x = \frac{2}{3} xy$
 $= \text{area of the parabola, and is the same as the rule.}$

Coroll. Every parabola is $= \frac{2}{3}$ of its circumscribing paral-
 lelogram.

EXAMPLES.

EXAMPLES.

1. What is the area of the parabola GVK, whose height VH is 12, and the base or double ordinate GK 16.



16
12
—
192
2
—
3)384

area required 128

2. The abscissa is 12, and the double ordinate of base 38: what is the area? *Ans.* 304.

PROBLEM X.

To find the area of the frustum of a parabolas.

RULE.

Divide the difference of the cubes of the two ends of the frustum by the difference of their squares, and this quotient multiplied by $\frac{2}{3}$ of the altitude, will give the area required.

* *Demon.* Let $D = GK$, $d = EI$, and $a = FH$.
Then by the nature of the curve $D^2 - d^2 : a :: D^3 : aD^2$
 $\frac{aD^2}{D^2 - d^2} = VH$, and $D^2 - d^2 : a :: d^3 : \frac{ad^2}{D^2 - d^2} = VF$.
And therefore $\frac{2}{3} \times \frac{aD^2}{D^2 - d^2} - \frac{2}{3} \times \frac{ad^2}{D^2 - d^2} = \frac{2}{3} a \times \frac{D^3 - d^3}{D^2 - d^2} = \text{area of the frustum.} \quad Q. E. D.$

EXAMPLES.

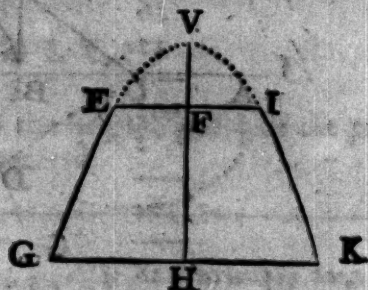
170 CONIC SECTIONS.

EXAMPLE.

1. In the parabolic frustum GEIK, the two parallel ends EI and GK are 6 and 10, and the altitude, or part of the abscissa FH, is 3: what is the area?

10
10

100 = square of GK
36 = square of EI
64



10
10

100
10

1000 = cube of GK.
216 = cube of EI.

64) 784 (12 $\frac{1}{2}$ = 12.25

64

2 = $\frac{2}{3}$ of FH

144
128

24.50 = area required.

16

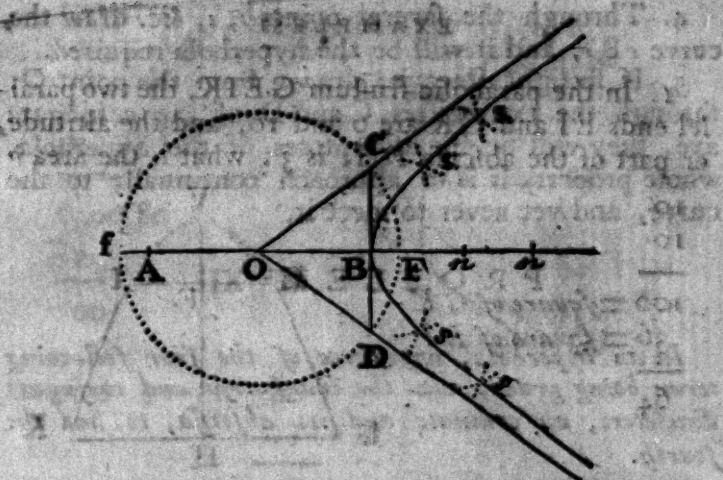
2. The greater end of the frustum is 24, the lesser end is 20, and their distance $5\frac{1}{2}$: what is the area?

Ans. 121.3.

PROBLEM XI.

To construct an hyperbola, the transverse and conjugate diameters being given.

Construction.



* *Construction.* 1. Make AB the transverse diameter, and CD perpendicular thereto, the conjugate.

2. Bisect AB in O, and from O with the radius OC, or OD, describe the circle DFC, cutting AB produced in F and f , and these points will be the two foci.

3. In AB produced take any number of points, n, n , &c. and from F and f , as centers, with the distances B n and A n , as radii, describe arcs cutting each other in s, s , &c.

* The sum of two lines drawn from the foci of an ellipse to any point in the curve, is equal to its transverse diameter. In like manner the difference of two lines drawn from the foci of an hyperbola to any point in the curve, is equal to its transverse diameter; as is shewn by the writers on conics. But the arcs intersecting each other in s, s , &c. are described from the foci f and F, and with the distances A n and B n , whose difference is AB, and therefore the points s, s , &c. are in the curve of an hyperbola.

4. Through

CONIC SECTIONS.

4. Through the several points s, s , &c. draw the curve sB , and it will be the hyperbola required.

5. If straight lines be drawn from the point O , through the extremities C and D of the conjugate axis, they will be the asymptotes of the hyperbola, whose property it is to approach continually to the curve, and yet never to meet it.

PROBLEM XII.

In an hyperbola, any three of the four following terms being given, viz. the transverse and conjugate diameters, an ordinate and its abscissa, to find the fourth.

CASE I.

To find the ordinate.

RULE.*

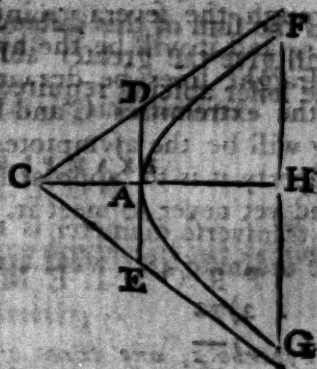
As the transverse diameter,
Is to the conjugate;
So is the square root of the product of the two abscissas,
To the ordinate required.

EXAMPLES.

1. In the hyperbola FAG , the transverse diameter twice AC is 120, the conjugate DE is 72, and the abscissa AH is 40: required the ordinate FH .

* Let t = transverse diameter, c = conjugate, x = abscissa, and y = ordinate. Then the general property of the curve is $t^2 : c^2 :: x \times t + x : y^2$; and from this analogy all the cases of this problem are deduced.

$$\begin{array}{r}
 120 = \text{twice } AC \\
 40 = AH \\
 \hline
 160 \\
 40 = AH \\
 \hline
 6400 (80) \\
 64 \\
 \hline
 00
 \end{array}$$



$$\begin{array}{r}
 120 : 72 :: 80 \\
 \hline
 72 \\
 \hline
 160 \\
 560 \\
 \hline
 12.0) 576.0
 \end{array}$$

48 = ordinate FH required.

2. The transverse diameter is 24, the conjugate 21, and the lesser abscissa 8 : what is its ordinate?

Ans. 14.

CASE II.

To find the abscissas.

RULE.

As the conjugate diameter,

Is to the transverse ;

So is the square root of the sum of the squares of the ordinate, and semi-conjugate,

To the distance between the ordinate and the center.

Then

Then the sum of this distance, and the semi-transverse will give the greater abscissa, and their difference the lesser abscissa required.

EXAMPLES:

The transverse diameter is 120, the conjugate 72, and the ordinate 48: what are the two abscissas?

$$2)72$$

$$\underline{36}$$

$$36$$

$$\underline{116}$$

$$108$$

$$1296 = \text{square of the semi-conjugate}$$

$$2304 = \text{square of the ordinate}$$

$$\underline{3600(60)}$$

$$36$$

$$00$$

$$\Delta 72 : 120 :: 60$$

$$\underline{120}$$

$$72)7200(100 = \text{diff. of the ordinate and } [center$$

$$\underline{72} \quad 60 = \text{semi-transverse}$$

$$00 \quad 160 = \text{greater abscissa}$$

$$\underline{40} = \text{lesser abscissa}$$

2. The transverse and conjugate diameters are 24 and 21: required the two abscissas to the ordinate 14.

Ans. 32 and 8.

CASE

CONIC SECTIONS.

115

CASE III.

To find the conjugate diameter.

RULE.

As the square root of the product of the two abscissas,
Is to the ordinate;
So is the transverse diameter,
To the conjugate.

EXAMPLES.

1. The transverse diameter is 120, the ordinate is 48, and the two abscissas are 160 and 40: required the conjugate.

160

40

6400 (80

64

00

As 80 : 48 :: 120

48

960

480

8.0)576.0

72 = conjugate required.

2. The transverse diameter is 24, the ordinate 14, and the abscissas 8 and 32: required the conjugate.

Ans. 21.

CASE

R U L E . *

1. To 19 times the transverse, add 21 times the parameter of the axe, and multiply the sum by the quotient of the abscissa divided by the transverse.

2. To 9 times the transverse add 21 times the parameter, and multiply the sum by the quotient of the abscissa divided by the transverse.

3. To each of the products, thus found, add 15 times the parameter, and divide the former by the latter; then this quotient being multiplied by the ordinate will give the length of the arc *nearly*.

E X A M P L E S .

1. In the hyperbola GAF, the transverse diameter twice AC is 80, the conjugate ED 60, the ordinate GH 10, and the abscissa AH 2.1637: required the length of the arc AG.

* *Demon.* Let t = semi-transverse axe, c = semi-conjugate, x = ordinate, and y = abscissa. Then will $y \times 1 + \frac{a^2}{6c^4} y^2 - \frac{a^4 + 4a^2c^2}{40c^8} y^4 + \frac{a^6 + 4a^4c^2 + 8a^2c^4}{112c^{12}} y^6$, &c. = length of the arc, as is shewn by the writers on fluxions.

But $x = \frac{a\sqrt{c^2+y^2}}{c} - a$, and $\frac{2c}{t}$ = parameter = p , by the nature of the curve. Consequently the rule is =

$$\frac{15p + \frac{19t + 21p}{t} x}{\frac{15p + 9t + 21p}{t} x} \times y = \frac{\frac{30c}{t} + \frac{19t + 42t}{t} \times \frac{a\sqrt{c^2+y^2}}{c} - a}{\frac{30c}{t} + \frac{9t + 42t}{t} \times \frac{a\sqrt{c^2+y^2}}{c} - a}$$

$\times y$.

And if this be thrown into a series it will be found to agree very nearly with the three first terms of the former, and therefore the rule is an approximation.

8,0)2.1637

$$\begin{array}{r} 80)2.1637 \\ \hline \end{array} \quad 80 : 60 :: 60$$

$$\begin{array}{r} .02704 \\ \hline \end{array}$$

$$80)360.0$$

$$45 = \text{parameter.}$$

$$21$$

$$45$$

$$90$$

$$945, \text{ and } 675 = 15 \text{ times } 45.$$

$$80 = \text{twice } AC.$$

$$19$$

$$720$$

$$80$$

$$1520$$

$$945$$

$$2465$$

$$.02704$$

$$9860$$

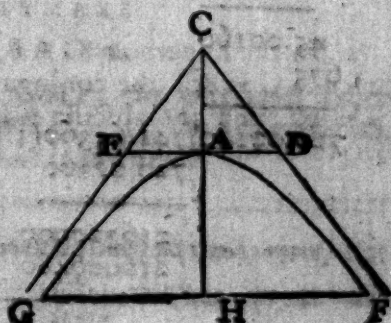
$$172550$$

$$4930$$

$$66.65360$$

$$675$$

$$741.6536$$



CONIC SECTIONS.

$$80 = \text{twice } AC.$$

9

720

945

1665

02704

6660

116550

3330

45.02160

675

720.02160) 741.65360 (1.03004

720 02160

216320000

216006480

313520000

288008640

25511360

1.03004

10 = GH.

10.30040 = length of the arc required.

PROBLEM XIV.

To find the area of an hyperbola.

RULE.

R U L E . *

1. To the product of the transverse and abscissa, add $\frac{1}{4}$ of the square of the abscissa, and multiply the square root of the sum by 21.

2. Add 4 times the square root of the product of the transverse and abscissa, to the last found product, and divide the sum by 75.

3. Divide 4 times the product of the conjugate and abscissa by the transverse, and this last quotient multiplied by the former will give the area required nearly.

E X A M P L E S .

1. In the hyperbola GAF, the transverse twice AC is 30, the conjugate ED is 18, and the abscissa or height AH is 10: what is the area.

* *Demon.* Let t = transverse diameter, c = conjugate, x = abscissa, y = ordinate, and $z = \frac{x^2}{2t+x}$. Then it is well

known that $4xy \times \frac{1}{3} - \frac{1}{6 \cdot 3 \cdot 5} z - \frac{1}{3 \cdot 5 \cdot 7} z^2 - \frac{1}{5 \cdot 7 \cdot 9} z^3$, &c. = area of the hyperbola.

But $\frac{ty}{\sqrt{tx+x^2}} = c$ = conjugate axe, by the nature of the hyperbola. Consequently the expression for the rule = $\frac{4cx}{t} \times \frac{21\sqrt{tx+\frac{1}{4}x^2}+4\sqrt{tx}}{75} = 4xy \times \frac{21\sqrt{tx+\frac{1}{4}x^2}+4\sqrt{tx}}{\sqrt{tx+x^2}}$.

And this thrown into a series will very nearly agree with the former, which shews the rule to be an approximation.

Q. E. I.

Rule 2. If $2Y$, $2y$ = bases, V , and v their distances from the center, and the other letters as before, then will $VY - vy - \frac{tc}{4} \times \text{hyp. log. of } \frac{tY+cV}{ty+cv}$ = area of the frustum of the hyperbola,

CONIC SECTIONS.

$$30 = 2AC.$$

$$10 = AH.$$

$$300$$

$$71.428571 = \frac{1}{3} \text{ of the square of } AH.$$

$$371.428571 (19.272$$

$$1 \quad 21$$

$$29)271$$

$$261$$

$$19272$$

$$38544$$

$$382)1042$$

$$764$$

$$404.712$$

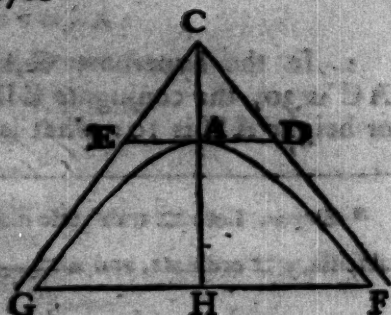
$$3847)27885$$

$$26929$$

$$38542)95671$$

$$77084$$

$$18587$$



$$30 = 2AC.$$

$$10 = AH.$$

$$300 (17.3205$$

$$1 \quad 4$$

$$27)200 \quad 69.2820$$

$$189$$

$$343)1100$$

$$1029$$

$$3462)7100$$

$$6924$$

$$346405)1760000$$

$$1732025$$

$$26975$$

$$04.712$$

CONIC SECTIONS.

133

$$\begin{array}{r} 404.712 \\ 69.282 \\ \hline \end{array}$$

$$75)473.994(6.3199$$

$$\begin{array}{r} 450 \\ \hline \end{array}$$

$$\begin{array}{r} 239 \\ \hline \end{array}$$

$$\begin{array}{r} 225 \\ \hline \end{array}$$

$$\begin{array}{r} 149 \\ \hline \end{array}$$

$$\begin{array}{r} 75 \\ \hline \end{array}$$

$$\begin{array}{r} 744 \\ \hline \end{array}$$

$$\begin{array}{r} 675 \\ \hline \end{array}$$

$$\begin{array}{r} 690 \\ \hline \end{array}$$

$$\begin{array}{r} 675 \\ \hline \end{array}$$

$$\begin{array}{r} 15 \\ \hline \end{array}$$

$$18 = ED.$$

$$10 = AH.$$

$$\begin{array}{r} 180 \\ \hline \end{array}$$

$$\begin{array}{r} 4 \\ \hline \end{array}$$

$$Twice AC = 3.0)72.0$$

$$\begin{array}{r} 24 \\ \hline \end{array}$$

$$\begin{array}{r} 6.3199 \\ \hline \end{array}$$

$$\begin{array}{r} 24 \\ \hline \end{array}$$

$$\begin{array}{r} 252796 \\ \hline \end{array}$$

$$\begin{array}{r} 126398 \\ \hline \end{array}$$

$$151.6776 = \text{area required.}$$

G 2

O 1

OF THE

MENSURATION

OF

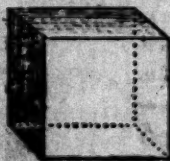
S O L I D S.

DEFINITIONS.

1. **T**HE *measure* of any solid body, is the whole capacity or content of that body, when considered under the triple dimensions of length, breadth, and thickness.

2. A *cube* whose side is one inch, one foot, or one yard, &c. is called the *measuring unit*; and the content or solidity of any figure, is computed by the number of those cubes contained in that figure.

3. A *cube* is a solid contained by six equal square sides.



4. A *parallelepipedon* is a solid contained by six quadrilateral sides, every opposite two of which are equal and parallel.

5. A



5. A *prism* is a solid whose ends are two equal, parallel, and similar plain figures, and its sides parallelograms.



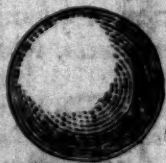
6. A *cylinder* is a solid described by the revolution of a right angled parallelogram about one of its sides which remains fixed.



7. A *pyramid* is a solid whose sides are all triangles meeting together in a point, and the base any plane figure whatever.



8. A *sphere* is a solid described by the revolution of a semi-circle about its diameter, which remains fixed.

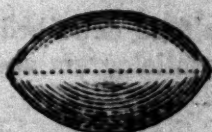


* The definition of a cone has been given already.

9. The center of the sphere is a point within the figure, every where equally distant from the convex surface of it.

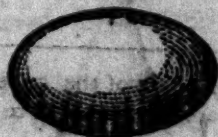
10. The diameter of the sphere is a straight line that passes through the center, and is terminated both ways by the convex superficies.

11. A *circular spindle* is a solid generated by the revolution of a segment of a circle about its chord, which remains quiescent.



12. A *spheroid* is a solid generated by the revolution of a semi-ellipsis about one of its diameters, which is considered as quiescent.

The spheroid is called *prolate*, when the revolution is made about the transverse diameter, and *oblate* when it is made about the conjugate diameter.



13. *Elliptic, parabolic, and hyperbolic spindles*, are generated in the same manner as the circular spindle, the double ordinate of the section being always fixed or quiescent.

14. *Parabolic and hyperbolic conoids*, are solids formed by the revolution of a semi-parabola or hyperbola about its transverse axis, which is considered as quiescent.



15. The

15. The *segment* of a pyramid, sphere, or any other solid, is a part cut off from the top by a plane parallel to the base of that figure.

15. A *frustum* or *trunk*, is the part that remains at the bottom after the segment is cut off.

17. The *zone* of a sphere, is that part which is intercepted between two parallel planes; and when those planes are equally distant from the center, it is called the middle zone of the sphere.

18. The height of a solid is a perpendicular, drawn from its vertex to the base or plane on which it is supposed to stand.

PROBLEM I.

To find the solidity of a cube.

RULE.

Multiply the side of the cube by itself, and that product again by the side, and it will give the solidity required.

Demon. Conceive the base of the cube to be divided into a number of little squares, each equal to the *superficial measuring unit*.

Then will those squares be the bases of a like number of small cubes, which are each equal to the *solid measuring unit*.

But the number of little squares contained in the base of the cube, are equal to the square of the side of that base, as has been shewn already.

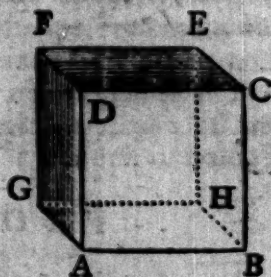
And consequently, the number of small cubes contained in the whole figure, must be equal to the square of the side of the base multiplied by the height of that figure; or, which is the same thing, the square of the side of the base multiplied by the base, is equal to the solidity. Q. E. D.

Note. The surface of the cube is equal to 6 times the square of its side.

EXAMPLES.

1. The side AB, or BC of the cube ABCDFGHE, is 25.5: what is the solidity?

$$\begin{array}{r}
 25.5 \\
 25.5 \\
 \hline
 1275 \\
 1275 \\
 510 \\
 \hline
 650.25 \\
 25.5 \\
 \hline
 325125 \\
 325125 \\
 \hline
 130050
 \end{array}$$



16581.375 the answer, or content of the cube.

2. The side of a cube is 15 inches: what is the solidity?

fo. in.

Ans. 1 11 5'.

3. What is the solidity of a cube whose side is 17.5?

Ans. 3.101 feet.

PROBLEM II.

To find the solidity of a parallelopipedon.

RULE.*

Multiply the length by the breadth, and that product again by the depth or altitude, and it will give the solidity required.

* The reason of this rule, as well as of the following ones for the prism and cylinder, is the same as that for the cube.

Note. The surface of the parallelopipedon is equal to the sum of the areas of each of its sides or ends.

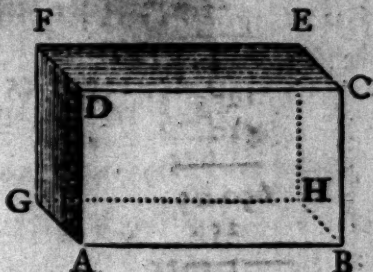
EXAM-

EXAMPLES.

1. Required the solidity of the parallelopipedon ABCDFEHG, whose length AB is 8 feet, its breadth FD $4\frac{1}{2}$ feet, and the depth or altitude AD $6\frac{1}{2}$ feet?

$$\begin{array}{l} 4.5 = \text{breadth } FD \\ 8 = \text{length } AB \end{array}$$

$$\begin{array}{r} 36.0 \\ 6.75 = \text{depth } AD \\ \hline 1800 \\ 2520 \\ \hline 2160 \end{array}$$



243.000 solid feet, the answer.

2. The length of a parallelopipedon is 15 feet, and each side of its square base 21 inches: what is the solidity?

Ans. 45.9 feet.

3. What is the solidity of a block of marble, whose length is 10 feet, its breadth $5\frac{1}{4}$ feet, and the depth $3\frac{1}{2}$ feet?

Ans. 201.25 feet.

PROBLEM III.

To find the solidity of a prism.

RULE.

Multiply the area of the base into the height, and the product will be the solidity.

EXAMPLES.

1. What is the solidity of the triangular prism ABCFED, whose length AB is 10 feet, and either of the equal sides BC, CD, or DB, of one of its equilateral ends BCD $2\frac{1}{2}$ feet?

G 5

2.5

2.5
 2.5
 2.5
 2.5
 $3.75 = \frac{1}{2}$ the sum of the sides.

$1.25, 125, 1.25 =$ three remainders,

1.25
 625

250
 125

1.5625
 1.25

78125
 31250

15625
 1.953125

3.75
 9765625

13671875
 5859375

7.32521875

4
 $47)332$

329
 $5406)34218$

32436
 $54123)178275$

162369
 15906



$2.7063 =$ area * of the base BCD.

$10 =$ length AB.

$27.0630 =$ solid feet, the answer.

* Note. If the areas of each of the sides and ends be calculated separately, their sum will be the surface of the prism.

2. What

2. What is the solidity of a triangular prism, whose length is 18 feet, and one side of the equilateral end $1\frac{1}{2}$ feet? *Ans.* 17.537 feet.

3. Required the solidity of a prism whose base is a hexagon, supposing each of the equal sides to be 1 foot 4 inches, and the length of the prism 15 feet. *Ans.* 69.282 feet.

PROBLEM IV.

To find the convex surface of a cylinder.

RULE.

Multiply the periphery of the end by the height of the cylinder, and the product will be the convex surface required.

EXAMPLES.

1. What is the convex surface of the right cylinder ABCD, whose length B is 20 feet, and the diameter of its base AB 2 feet?

$$\begin{array}{r} 3.14159 \\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 6.28318 = \text{periphery of the base AB.} \\ 20 = \text{height BC.} \\ \hline \end{array}$$

$$125.66360 = \text{square feet, the answer.}$$



* *Demon.* If the periphery of the base be conceived to move, in a direction parallel to itself, it will generate the convex superficies of the cylinder; and therefore the said periphery being multiplied by the length of the cylinder, will be equal to that superficies. Q. E. D.

Note. If twice the area of either of the ends be added to the convex surface, it will give the whole surface of the cylinder.

G 6

2. What

2. What is the convex surface of a right cylinder, the diameter of whose base is 30 inches, and the length 60 inches? *Ans. 5654.88 inches.*

3. Required the convex superficies of a right cylinder, whose circumference is 8 feet 4 inches, and its length 14 feet. *Ans. 116.666 &c.*

PROBLEM V.

To find the solidity of a cylinder.

RULE.*

Multiply the area of the base by the height, and the product will be the solidity.

EXAM-

* The four following cases contain all the rules for finding the superficies, and solidities of cylindric ungulas.

I. *When the section is parallel to the axis of the cylinder.*



Rule. 1. Multiply the length of the arc line of the base by the height of the cylinder, and the product will be the curve surface.

2. Multiply the area of the base by the height of the cylinder, and the product will be the solidity.

II. *When the section passes obliquely through the opposite sides of the cylinder.*



Rule,

EXAMPLES.

1. What is the solidity of the cylinder ABCD, the diameter of whose base AB is 30 inches, and the height BC 50 inches?

Rule. 1. Multiply the circumference of the base of the cylinder by half the sum of the greatest and least lengths of the ungula, and the product will be the *curve surface*.

2. Multiply the area of the base of the cylinder by half the sum of the greatest and least lengths of the ungula, and the product will be the *solidity*.

III. When the section passes through the base of the cylinder, and one of its sides.



Rule. 1. Multiply the sine of half the arc of the base by the diameter of the cylinder, and from this product subtract the product of the arc and cosine.

2. Multiply the difference, thus found, by the quotient of the height divided by the versed sine, and the product will be the *curve surface*.

1. From $\frac{2}{3}$ of the cube of the right sine of half the arc of the base, subtract the product of the area of the base and the cosine of the said half arc.

2. Multiply the difference, thus found, by the quotient arising from the height divided by the versed sine, and the product will be the *solidity*.

IV. When the section passes obliquely through both ends of the cylinder.



Rule.

$$.7854$$

$$900 = \text{square of } AB$$

$$706.8600 = \text{area of the base } AB$$

$$50$$

$$1728) 35343.0000 (20.4531 = \text{solidity required.}$$

$$3456$$

$$7830$$

$$6912$$

$$9180$$

$$8640$$

$$5400$$

$$5184$$

$$2160$$

$$1728$$

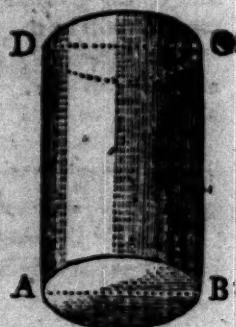
$$432$$

2. What is the solidity of a cylinder whose height is 5 feet, and the diameter of the end 2 feet?

Ans. 15.708 feet.

3. What is the solidity of a cylinder whose height is 20 feet, and the circumference of its base 20 feet also?

Ans. 636.61828 feet.



Rule. 1. Conceive the section to be continued, till it meets the side of the cylinder produced; then say, as the difference of the versed sines of half the arcs of the two ends of the ungula, is to the versed sine of half the arc of the lesser end, so is the height of the cylinder to the part of the side produced.

2. Find the surface of each of the ungulas, thus formed, by case 3d, and their difference will be the *surface required*.

3. In like manner find the solidities of each of the ungulas, and their difference will be the *solidity required*.

4. The

4. The circumference of the base of an oblique cylinder is 20 feet, and the perpendicular height 19.318 : what is the solidity? *Ans.* 614.926 feet.

PROBLEM VI.

To find the convex surface of a right cone.

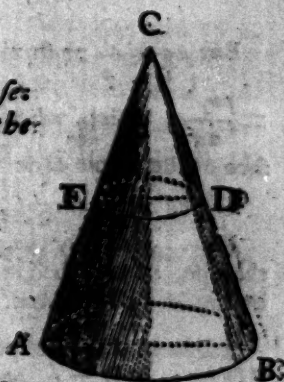
RULE.*

Multiply the circumference of the base by the slant height, or the length of the side, and half the product will be the surface required.

EXAMPLES.

1. The diameter of the base AB is 3 feet, and the slant height AC or BC 15 feet : required the convex surface of the cone ACB.

$$\begin{array}{r}
 3.1416 \\
 \times 3 \\
 \hline
 9.4248 = \text{circumference of the base} \\
 \times 15 = \text{slant height BC.} \\
 \hline
 471240 \\
 94248 \\
 \hline
 2) 141.3720 \\
 \hline
 70.686 = \text{convex surface re-} \\
 \text{quired.}
 \end{array}$$



2. The

* *Demon.* Let $AB = a$, $BC = b$, $3.1416 = p$, and $ED = y$.
Then $a : b :: y : \frac{by}{a} = DC$; and $py =$ circumference of the circle ED.

But

2. The diameter of a right cone is 4.5 feet, and the slant height 20 feet: required the convex surface.

Ans. 141.372 feet.

3. The circumference of the base is 10.75, and the slant height 18.25: what is the convex surface?

Ans. 98.09375.

PROBLEM VII.

To find the convex surface of the frustum of a right cone.

R U L E. *

Multiply the sum of the perimeters of the two ends, by the slant height of the frustum, and half the product will be the surface required.

EXAM-

But $py \propto \frac{by^2}{a}$ = fluxion of the surface of CED, and its fluent $= \frac{pby^2}{2a}$; which, when $y = a$, becomes $\frac{pba}{2}$ = convex surface of the whole cone. Q. E. D.

To find the surface of a right pyramid.

Rule. Multiply the perimeter of the base by the length of the side, or slant height of the cone, and half the product will be the surface required.

* *Demon.* Let the perimeter of the circle $AB = P$, that of $ED = p$, $BD = b$, and the rest as in the last problem.

Then $P : p :: b (BC) : CD$, and, by division, $P - p :: b - CD (b) : CD = \frac{pb}{P - p}$; but $P \times b + \frac{pb^2}{P - p}$ = twice the convex surface of the whole cone, by the last rule; and also $p \times \frac{pb}{P - p}$ = twice the convex surface of the part ECD. There-

for

EXAMPLES:

1. In the frustum ABED, the circumferences of the two ends AB and DE are 22.5 and 15.75 respectively, and the slant height BD is 26: what is the convex surface?

22.5

15.75

38.25 = sum of the perimeters.

26 = slant height BD

22950

7650

2)994.50

497.25 = convex surface.



2. What is the convex surface of the frustum of a right cone, the circumference of the greater end being 30 feet, that of the lesser end 10 feet, and the length of the slant side 20 feet? *Ans. 400 feet.*

3. What is the convex surface of the frustum of a right cone, the diameters of the ends being 8 and 4 feet, and the length of the slant side 20 feet?

Ans. 376.992 feet.

fore $P \times b + \frac{pb}{P-p} - p \times \frac{pb}{P-p} = bP + P - p \times \frac{pb}{P-p} = bP + bp = P + p \times b =$ twice the convex surface of the frustum ABDE, and the half thereof is $\frac{P+p \times b}{2}$, the same as the rule. Q. E. D.

To find the surface of the frustum of a right pyramid.

Rule. Multiply the sum of the perimeters of the end by the slant height, and half the product will be the surface required.

P R Q.

PROBLEM VIII.

To find the solidity of a cone or pyramid.

R U L E. *

Multiply the area of the base by $\frac{1}{3}$ of the height, and the product will be the solidity.

E X A M P L E S.

1. Required the solidity of the cone ACB, whose diameter AB is 20, and its perpendicular height CS 24.

* *Demon.* Let CS = a , C $\dot{x}$ = x , and A = area of the base of the cone ACB.

Then a^2 (CS 2) : x^2 (C $\dot{x}$ 2) : AB 2 : ED 2 (by sim. Δ) :: A : $\frac{Ax^2}{a^2}$, (= area of the circle ED) because all circles are as the squares of their diameters.

But $\frac{Ax^2}{a^2} \times \dot{x}$ = fluxion of the cone ECD, and its fluent = $\frac{Ax^3}{3a^2}$; which, when $x = a$ becomes $\frac{Aa}{3} = A \times \frac{a}{3}$ for the solidity of the whole cone. Q. E. D.

In the pyramid CEDB it will be a^2 (CS) : x^2 (C $\dot{x}$ 2) :: CE 2 : C $\dot{e}$ 2 :: ED 2 : eo 2 (by sim. Δ) :: A (area of the base EB) : $\frac{Ax^2}{a^2}$ (area of the polygon eb) because all similar figures are as the squares of their like sides.

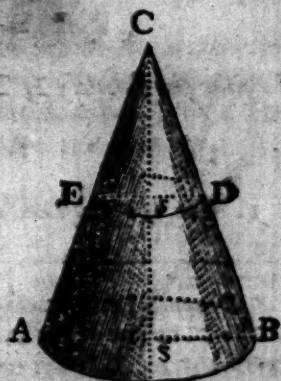
But $\frac{Ax^2}{a^2} \times \dot{x}$ = fluxion of the pyramid Ceob, and its correct fluent = $A \times \frac{a}{3}$ the same as in the cone; and this rule is general, let the figure of the base be what it will.

.7854

400 = square of AB

314.1600 = area of the base
8 = $\frac{1}{3}$ of CS

2513.2800 = solidity required.



2. Required the solidity of the hexagonal pyramid ECBD, each of the equal sides of its base being 40, and the perpendicular height CS 60.

[the side is 1.

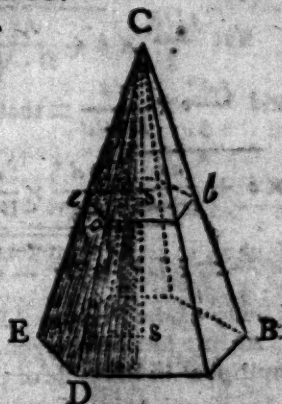
2.598076 = multiplier when
1600 = square of CE

1558845600

2598076

4156.921600 = area of the base
20

83138.432000 = solidity requir.



3. Required the solidity of a triangular pyramid, whose height is 30, and each side of the base 3.

Ans. 38.97117.

4. Required the solidity of a square pyramid, each side of whose base is 30, and the perpendicular height 20.

Ans. 6000.

5. What is the solidity of a cone, the diameter of whose base is 18 inches, and its altitude 15 feet?

Ans. 8.83575 feet.

6. If the circumference of the base of a cone be 40 feet, and the height 50 feet; what is the solidity?

Ans. 2120 feet.

P R Q.

PROBLEM IX.

To find the solidity of the frustum of a cone or pyramid.

RULE.*

1. For the frustum of a cone.

* *Demon.* Let D = diameter AB , d = ED , p = .7854, and b = S = height of the frustum of the cone $ABDE$, (see the last figure.)

Then $D : d :: CS : Cs$, and $D - d : d :: CS - Cs (b) : \frac{db}{D - d} = Cs$.

But $\frac{pD^2}{3} \times b + \frac{db}{D - d}$ = solidity of the whole cone ACB ,
and $\frac{pd^2}{3} \times \frac{db}{D - d}$ = that of the cone ECD . Therefore $\frac{pD^2}{3}$
 $\times b + \frac{db}{D - d} - \frac{pd^2}{3} \times \frac{db}{D - d} = D^2 \times b + \frac{db}{D - d} - d^2 \times \frac{db}{D - d}$
 $\times \frac{p}{3} = bD^2 + D^2 - d^2 \times \frac{db}{D - d} \times \frac{p}{3} = \frac{D^3 - d^3}{D - d} \times \frac{bp}{3} =$
solidity of the frustum $ABDE$. Q. E. D.

Again, for the polygon, let S = ED , s = ed , and m = proper multiplier in the table of polygons; then $S : s :: CS : Cs$, and
 $S - s : s :: CS - Cs (b) : \frac{bs}{S - s}$.

But mS^2 and ms^2 are the areas of polygons whose sides are
 S and s respectively. And therefore $\frac{mS^2}{3} \times b + \frac{bs}{S - s} - \frac{ms^2}{3} +$
 $\frac{bs}{S - s} = mS^2 + mS^2 - ms^2 \times \frac{bs}{S - s} \times \frac{b}{3} = mS^2 + msS +$
 $\frac{ms^2}{3} \times \frac{b}{3}$ = solidity of the frustum $EDBb$, and is the same as
the rule.

Divide

Divide the difference of the cubes of the diameters of the two ends, by the difference of the diameters, and this quotient being multiplied by .7854 and again by $\frac{1}{3}$ of the height will give the solidity.

2. For the frustum of a pyramid.

To the areas of the two ends of the frustum add the square root of their product, and this sum being multiplied by $\frac{1}{3}$ of the height will give the solidity.

1. What is the solidity of the frustum of the cone EABD, the diameter of whose greater end AB is 5 feet, that of the lesser end ED 2 feet, and the perpendicular height S 9 feet?

5	3	
5	3	
—	—	
25	9	5
—	—	—
125	27	2
		—
125 = cube of AB. 27 = cube of ED. 2 = difference of AB & ED.		
2) 98	.7854	
—	147	
49	—	
3 = $\frac{1}{3}$ height S	54978	
—	31416	
147	7854	
	—	



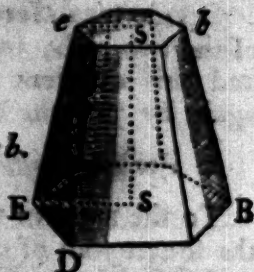
115.4538 = solidity required.

2. What is the solidity of the frustum EDBb, of an hexagonal pyramid, the side ED of whose greater end is 4 feet, that ed of the lesser end 3 feet, and the height S 9 feet?

2.598076

2.598076 = tabular multiplier.
 9 = square of ed .

23.382684 = area of the polygon eb .



2.598076 = tabular multiplier.
 16 = square of ED .

15588456
 2598076

41.569216 = area of the polygon EB .
 23.382684

166276864
 332553728
 249415296
 83138432
 332553728
 124707648
 124707648
 83138432

971.999841855744

971.999841855744

971.999841855744 (31.176911 = square root of
 9 the product of the areas of
 — the two ends.

61)71
 61

621)1099
 621

41.569216
 23.382684
 31.176911

6227)47898
 43589

96.128811

$3 = \frac{1}{3}$ of the height 8

62346)430941
 374076

288.386433 = solidity required.

623529)5686585
 5611761

6235381)7282457
 6235381

62353821)104707644
 62353821

42353823

3. What is the solidity of the frustum of a cone, the diameter of the greater end being 4 feet, that of the lesser end 2 feet, and the altitude 9 feet?

Ans. 65.9736.

4. What is the solidity of the frustum of a cone, the circumference of the greater end being 40, that of the lesser end 20, and the length or height 50?

Ans. 3713.64.

5. What is the solidity of the frustum of a square pyramid, one side of the greater end being 18 inches, that of the lesser end 15 inches, and the height 60 inches?

Ans. 16380 inches.

6. What

6. What is the solidity of the frustum of an hexagonal pyramid, the side of whose greater end is 3 feet, that of the lesser end 2 feet, and the length 12 feet?

Ans. 197.453776 feet.

PROBLEM XI.

To find the solidity of a cuneus or wedge.

RULE.*

Multiply the sum of twice the length of the base and the length of the edge, by the product of the height of the wedge and the breadth of the base, and $\frac{1}{6}$ of the last product will be the solidity.

EXAMPLES.

1. How many solid feet are there in a wedge, whose base is 5 feet 4 inches long, and 9 inches broad, the length of the edge being 3 feet 6 inches, and the perpendicular height 2 feet 4 inches?

* *Demon.* When the length of the base is equal to that of the edge, the wedge is evidently equal to half a prism of the same base and altitude.

And according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism, by a pyramid of the same height and breadth at the base with the wedge, and the length of whose base is equal to the difference of the lengths of the edge and base of the wedge.

Therefore, let the length of the base $BC=L$; the length of the edge $EF=l$; the breadth of the base $BA=b$; and the height of the wedge $EP=b$; and we shall have by the former

$$\text{rules } \frac{blb}{2} \pm bb \times \frac{\pm L \mp l}{3} = \frac{blb}{2} + bb \times \frac{L-l}{3} = bb \times \frac{3/4 L + 2l - 2l}{6} = bb \times \frac{3L + l}{6}. \quad Q. E. D.$$

64
2

128 = twice the length of the base BC.
42 = length of the edge EF.

170 = sum.

28 = height of the wedge EP.
9 = breadth of the base AB.

252 = product.
170

17640
252

6)42840

1728)7140(4.1319 = solid feet, the answer.

6912

2280

1728

5520

5184

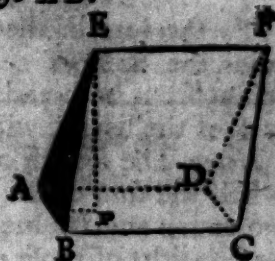
3360

1728

16320

15552

768



2. The length and breadth of the base of a wedge are 35 and 15 inches, and the length of the edge is

H

55

55 inches: what is the solidity, supposing the perpendicular height to be 17.14508?

Ans. 3.1006 feet.

PROBLEM XII.

To find the solidity of a prismoid.

RULE.*

To the sum of the areas of the two ends add four times the area of a section parallel to and equally distant from both ends, and this last sum multiplied by $\frac{1}{6}$ of the height will give the solidity.

Note. The length of the middle rectangle is equal to half the sum of the lengths of the rectangles of the two ends, and its breadth equal to half the sum of the breadths of those rectangles.

EXAMPLES.

1. How many solid feet are there in a tree whose ends are rectangles, the length and breadth of the

* *Demon.* The rectangular prismoid is evidently composed of two wedges, whose heights are equal to the height of the prismoid, and their bases its two ends. Wherefore, by the

last problem, its solidity will be $= \frac{2L+l}{3} \times \frac{B+b}{2}$, which, by putting $M = \frac{L+l}{2}$, and $m = \frac{B+b}{2}$, becomes

$\frac{BL+bl+4Mm}{6} \times \frac{h}{1}$; which is the rule, as was to be shewn.

The solidities of the two parts, commonly called the ungulus, or hoofs, into which the frustum of a rectangular pyramid is divided, may be found by the two last rules, as they are only composed of wedges and prismoids.

A very elegant demonstration of this rule for the prismoid may be seen in *Simpson's Fluxions*, page 179, 2d edit.

one

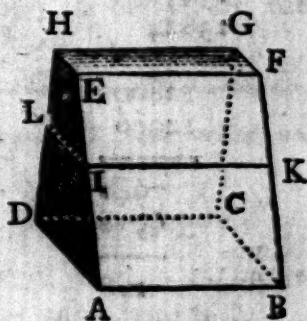
one being 14 and 12 inches, and the corresponding sides of the other 6 and 4 inches; the perpendicular length being $30\frac{1}{2}$ feet?

$$\begin{array}{r} 14 \\ 12 \\ \hline \end{array}$$

168 = area of the base ABCD

$$\begin{array}{r} 6 \\ 4 \\ \hline \end{array}$$

24 = area of the top EFGH



$$\begin{array}{r} 14 \\ 6 \\ \hline \end{array}$$

$$2)20$$

10 = length of the mid. rect. IK 8 = breadth IL

$$\begin{array}{r} 8 \\ 4 \\ \hline \end{array}$$

320 = 4 times the area of the middle rectangle.

$$\begin{array}{r} 168 \\ 24 \\ \hline \end{array}$$

512 = $\frac{1}{2}$ of the length.

$$\begin{array}{r} 512 \\ 3072 \\ \hline 31232 \end{array}$$

1728) 31232 (18.074 *Solid feet, the answer.*

1728

13952

13824

12800

12096

7040

6912

128

2. What is the solid content of a prismoid, whose greater end measures 12 inches by 8, the lesser end 8 inches by 6, and the length, or height, 60 inches? *Ans. 2.453 feet.*

3. What is the capacity of a coal waggon whose inside dimensions are as follows: at the top, the length is $81\frac{1}{2}$, and breadth 55 inches; at the bottom, the length is 41, and breadth $29\frac{1}{2}$ inches; and the perpendicular depth is $47\frac{1}{2}$ inches? *Ans. 126752.0625 cubic inches; which is nearly equal to a chaldron of coals.*

PROBLEM XIII.

To find the convex surface of a sphere.

RULE.*

Multiply

* *Demon.* Put the diameter $BG=d$, $BA=x$, $AC=y$, $BC=z$; and 3.14159 , &c. $=p$.

Then,

Multiply the diameter of the sphere by its circumference, and the product will be the convex superficies required.

Note. The curve surface of any zone or segment will also be found by multiplying its height by the whole circumference of the sphere.

Then, since the triangles AOC and CED are similar, we shall have $CA(y) : CO\left(\frac{d}{2}\right) :: CE(\dot{x}) : CD(\dot{z}) = \frac{d\dot{x}}{2y}$. But $2py\dot{z}$ is the general expression for the fluxion of any surface; and therefore by substituting $\frac{d\dot{x}}{2y}$ for its equal $\dot{z}$, the fluxion will become $p d\dot{x}$; and consequently $p d\dot{x} =$ surface of any segment of a sphere whose height is x , and $p d\dot{x} =$ that of the whole sphere. Q. E. D.

Coroll. The surface of the sphere is equal to the curve surface of its circumscribing cylinder.

The two following rules may sometimes be found useful.

1. To find the lunar surface included between two great circles of a sphere.

R U L E.

Multiply the diameter into the breadth of the surface in the middle, and the product will be the superficies required.

Or,

As one right angle is to a great circle of the sphere;

So is the angle made by the two great circles,

To the surface included by them.

2. To find the area of a spherical triangle, or the surface included by the intersecting arcs of three great circles of the sphere.

R U L E.

As two right angles, or 180° ,

Is to a great circle of the sphere;

So is the excess of the three angles above two right angles,

To the area of the triangle.

H 3

E X A M -

EXAMPLES.

1. What is the convex superficies of a globe BCG whose diameter BG is 17 inches?

$$\begin{array}{r} 3.14159 \\ 17 \\ \hline \end{array}$$

$$\begin{array}{r} 2199113 \\ 314159 \\ \hline \end{array}$$

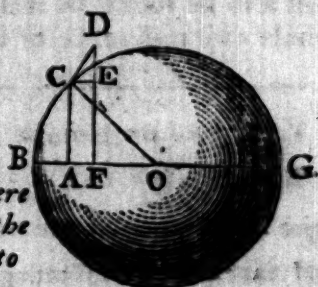
$$53.40703 = \text{circumf. of the sphere}$$

$$17 = \text{diameter of ditto}$$

$$\begin{array}{r} 37384921 \\ 5340703 \\ \hline \end{array}$$

$$5340703$$

$$907.91951 = \text{square inches, the answer.}$$



2. What is the convex superficies of a sphere whose diameter is $1\frac{1}{3}$ feet, and the circumference 4.1888 feet?

Ans. 5.58506 feet.

3. If the diameter, or axis of the earth, be $7957\frac{1}{2}$ miles; what is the whole surface, supposing it to be a perfect sphere?

Ans. 198943750 square miles.

4. The diameter of a sphere is 21 inches; what is the convex superficies of that segment of it whose height is $4\frac{1}{2}$ inches?

Ans. 296.8812 inches.

5. What is the convex surface of a spherical zone, whose breadth is 4 inches, and cut from a sphere of 25 inches diameter?

Ans. 314.16 inches.

PROBLEM XIV.

To find the solidity of a sphere or globe.

RULE.

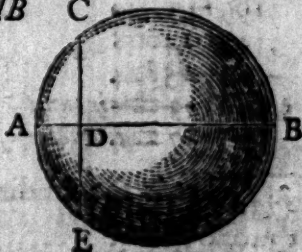
R U L E . *

Multiply the cube of the diameter by .5236, and the product will be the solidity.

E X A M P L E S .

1. What is the solidity of the sphere AEBC, whose diameter AB is 17 inches?

17 = diameter AB C



$$\begin{array}{r}
 17 \\
 \hline
 119 \\
 17 \\
 \hline
 289 \\
 17 \\
 \hline
 2023 \\
 289 \\
 \hline
 4913 = \text{cube of the diameter AB.} \\
 .5236 \\
 \hline
 29478 \\
 14739 \\
 9826 \\
 24565 \\
 \hline
 2572.4468
 \end{array}$$

* *Demon.* Put $AD = x$, $CD = y$, the diameter $AB = d$, and $p = 3.14159$.

Then, by the property of the circle, $dx - x^2 = y^2$. But the general expression for the fluxion of any solid is $py^2\dot{x}$; and therefore by writing $dx - x^2$ for its equal y^2 , we shall have $p\dot{x} \times dx - x^2 = pdx\dot{x} - px^2\dot{x}$. The fluent of which is $\frac{pdx^2}{2} - \frac{px^3}{3} = \frac{3pdx^2 - 2px^3}{6}$ = content of the segment CAE.

H 4

And

1728) 2572.4468 (1.4308 *solid feet, the answer.*

1728

—

7444

6912

—

5324

5184

—

14068

13824

—

244

2. What is the solidity of a sphere whose diameter is $1\frac{1}{2}$ feet?

Ans. 1.2411 feet.

3. What is the solidity of the earth, supposing it to be perfectly spherical, and its diameter 7957 miles?

Ans. 26385743776 miles.

PROBLEM XV.

To find the solidity of the segment of a sphere.

RULE.

To

And if d be substituted for x it will become $\frac{3pd^3 - 2pd^2}{6} =$

$\frac{pd^3}{6} = d^3 \times .5236$, which is the rule.

Coroll. A globe is equal to $\frac{2}{3}$ of its circumscribing cylinder.

* *Demon.* Let r = radius of the base of the segment, b = height of the segment, and the other letters as before.

Then

To three times the square of the radius of its base add the square of its height; and this sum multiplied by the height, and the product again by .5236 will give the solidity.

EXAMPLES.

1. The radius Cn of the base of the segment CAD is 7 inches, and the height An 4 inches: what is its solidity?

$$\begin{array}{r}
 7 \\
 7 \\
 \hline
 49 \\
 3 \\
 \hline
 147 = 3 \text{ times the square of } \\
 16 = \text{square of } An. \\
 \hline
 163 \\
 4 = \text{height } An. \\
 \hline
 652 \\
 .5236 \\
 \hline
 3912 \\
 1956. \\
 1304 \\
 \hline
 3260 \\
 \hline
 341.3872 = \text{solid inches, the answer.}
 \end{array}$$

Then will $3db^2 - 2b^3 \times \frac{p}{6} = \text{solidity of the segment, as is shewn in the last problem.}$

But since $\frac{r^2 + b^2}{b} = d$, by the property of the circle, we shall have $\frac{3r^2b^2 + 3b^4}{b} - 2b^3 \times \frac{p}{6} = \frac{3r^2 + b^2}{3} \times \frac{ph}{6} = \text{solidity of the segment, which is the same as the rule.}$

H. 5

2. What

2. What is the solidity of the segment of a sphere, the diameter of whose base is 10, and its height $4\frac{1}{2}$?

Ans. 224.95165.

3. What is the solidity of a spherical segment, the diameter of its base being 17.23368, and its height 4.5?

Ans. 572.5566.

PROBLEM XVI.

To find the solidity of a frustum or zone of a sphere.

RULE.*

To the sum of the squares of the radii of the two ends add $\frac{1}{3}$ of the square of their distance, or the breadth of the zone, and this sum multiplied by the said breadth, and the product again by 1.5708 will give the solidity.

EXAMPLES.

1. What is the solid content of the zone ABCD, whose greater diameter AB is 20 inches, the lesser diameter CD 15 inches, and the distance *nm* of the two ends 10 inches?

* *Demon.* The difference between two segments of a sphere whose heights are *H* and *b*, and the radii of whose bases are

R and *r*, will, by the last problem $= \frac{1}{6} \times 3R^2H + H^3 -$

$3r^2b - b^3 =$ zone whose height is *H* - *b*. And therefore by putting *a* for the altitude of the frustum, and exterminating

H and *b* by means of the two equations $\frac{R^2 + H^2}{H} = \frac{r^2 + b^2}{b}$

and *H* - *b* = *a*, we shall have $R^2 + r^2 + \frac{a^2}{3} \times \frac{pa}{2}$, which is the rule.

$$100 = \text{square of } A m.$$

$$56.25 = \text{square of } D n.$$

$$156.25$$

$$33.33 = \frac{1}{3} \text{ square of } n m.$$

$$189.58$$

$$10 = \text{breadth } n m.$$

$$1895.80$$

$$1.5708$$

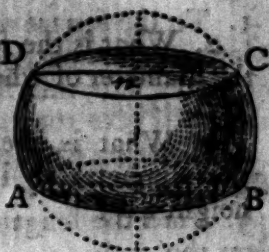
$$1516640$$

$$13270600$$

$$947900$$

$$189580$$

$$2977.922640 = \text{solid inches, the answer.}$$



2. What is the solid content of a zone, whose greater diameter is 24 inches, the lesser diameter 20 inches, and the distance of the ends 4 inches?

Ans. 1566.6112 inches.

3. Required the solidity of the middle zone of a sphere, whose top and bottom diameters are each 3 feet, and the breadth of the zone 4 feet.

Ans. 61.7848 feet.

PROBLEM XVII.

To find the surface of a circular spindle, the length and breadth, or middle diameter, being given.

RULE.*

* *Demon.* Put $x = \text{arc } C n$, $x = \text{its sine } a m$, $r = \text{radius } C$, $c = \text{central distance } o r$, and $p = 3.14159$, &c.

H 6

Then

1. To the square of half the length of the spindle, or longest diameter, add the square of half the middle diameter, and this sum divided by the middle diameter will give the *radius of the circle*.

2. Take half the middle diameter from the radius of the circle, and it will give the *central distance*.

3. Find the length of the revolving arc by problem the 10th.

4. From the product of the longest diameter and the radius of the revolving arc, subtract the product of the said arc and the central distance, and this remainder multiplied by 6.2832 will give the surface required.

EXAMPLES.

1. What is the superficial content of the circular spindle ADBC, whose length AB is 48, and the middle diameter CD 36?

Then $\sqrt{r^2 - x^2} (om) : r :: \dot{x} : \dot{z}$, and $\dot{z} \sqrt{r^2 - x^2} = r \dot{x}$, by the property of the circle, as is shewn in the demonstration to problem 13. But $2p \dot{z} \times ns = 2p \dot{z} \sqrt{r^2 - x^2} - c = 2p \times \dot{z} \sqrt{r^2 - x^2} - c \dot{z}$ is the general expression for the fluxion, and therefore $2p \times r \dot{x} - c \dot{z} = \text{fluxion of half the frustum } CncD$, and $2p \times rn - cz = 2p \times r \times ac - c \times Cn = \text{to its surface.}$

Q. E. D.

Coroll. 1. When $ca = cA$, the rule becomes $2p \times r \times cA - c \times CA$ for the surface of half the spindle, and $2p \times r \times AB - c \times ACB$ for that of the whole spindle.

Coroll. 2. If from the surface of the semi-spindle there be taken that of the frustum, there will remain $2p \times r \times aA - c \times Aa$ for the segment aAc ; so that the rule is general.

1. To the square of half the length of the chord
or longest distance, add the square of half the middle
distance, and this sum divided by the middle
distance will give the radius of the circle.
2. Find the length of the chord, and from the
middle distance, subtract the square of the
radius of the circle, and the remainder
of the first and the square of the
radius of the circle, and the remainder
required.



$$18 = Ce, \text{ or } \frac{1}{2} CD.$$

$$\begin{array}{r} 144 \\ 18 \end{array}$$

$$324 = \text{square of } Ce.$$

$$24 = Ae, \text{ or } \frac{1}{2} \text{ of } AB.$$

$$\begin{array}{r} 24 \\ 96 \\ 48 \end{array}$$

$$576 = \text{square of } Ae.$$

$$324$$

$$36) 900 (25 = \text{radius o } C.$$

$$72 \times 18 = Ce.$$

$$180 \quad 7 = \text{central distance o } c.$$

$$180$$

$$\begin{array}{r}
 18 \\
 2 \\
 \hline
 3 \overline{) 36} \\
 \hline
 12 = \frac{2}{3} \text{ of } Cc.
 \end{array}$$

$$\begin{array}{r}
 18 = Cc. \\
 41 \\
 \hline
 18 \\
 72 \\
 \hline
 5.0 \overline{) 73.8}
 \end{array}$$

$$\begin{array}{r}
 14.76 = \frac{41}{8} \text{ of } Cc. \\
 50 = \text{diameter.}
 \end{array}$$

$$\begin{array}{r}
 35.24 \overline{) 12.000} (.34052 \\
 10.572
 \end{array}$$

$$\begin{array}{r}
 14280 \\
 14096
 \end{array}$$

$$\begin{array}{r}
 18400 \\
 17620
 \end{array}$$

$$\begin{array}{r}
 7800 \\
 7048
 \end{array}$$

$$752$$

$$\begin{array}{r}
 1.34052 \\
 -48
 \end{array}$$

$$\begin{array}{r}
 1072416 \\
 536208
 \end{array}$$

$$\begin{array}{r}
 64.34496 = \text{length of the arc } ACB. \\
 7 = \text{central distance.}
 \end{array}$$

$$450.41472$$

48 = longest diameter AB.

25 = radius aC.

—
24096
—

1200

450.41472
—

749.58528

6.2832
—

149917056

224875584

599668224

149917056

449751168
—

4709.794231296 = superficies required.

2. What is the superficial content of a circular spindle whose length is 48, and its middle diameter 30?

Ans. 3270.5335.

PROBLEM XVIII.

To find the solidity of a circular spindle, the length and middle diameter being given.

RULE.*

1. Find

* Demon. Put $ac = x$, $os = c$, $r = \text{radius}$, and $p = 3.14159$, &c.

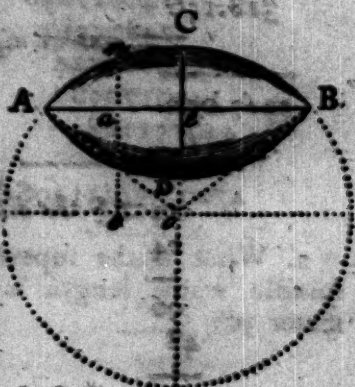
Then will the fluxion of the solid be $= p\dot{x} \times na^2 = \frac{p\dot{x} \times}{nc - a^2} = p\dot{x} \times nc^2 - c \times 2nc - c = p\dot{x} \times nc^2 - c \times 2an + c = p\dot{x}$

1. Find the area of the generating circular segment by problem the 13th, and the radius and central distance as in the last problem.

2. From $\frac{1}{2}$ of the cube of half the length of the spindle subtract the product of the central distance and half the area of the generating segment, and this remainder multiplied by 12.5664 will give the solidity.

EXAMPLES.

1. The longest diameter AB of the circular spindle ADBC is 48, and the middle diameter CD 36: what is the solidity of the spindle?



48 = diameter AB.

7 = central distance to

2) 336.

angle AOB

168 = area of the tri-

$= p \times r^2 - x^2 - c^2 - 2c \times ax$; whose fluent is $= p \times$

$\frac{p^2}{3} - \frac{x^2}{3} - c^2 - \frac{2c}{x} \times anC$ $= p \times Ae^2 - \frac{x^2}{3} \times x - 2c \times$

anC = frustum generated by anC . And when $x = Ae$, we have $\frac{1}{3} Ae^3 - c \times ACe \times 2p$ for the half spindle CAD, and $\frac{2}{3} Ae^3 - c \times ACe \times 4p$ = whole spindle ADBC.

Q. E. D.

Coroll. If the frustum be taken from the half spindle there will remain $p \times Ae^2 \times Ae - \frac{1}{3} \times Ae^3 - c^2 - 2c \times Ana = p \times \frac{1}{3} Ae^3 \times \frac{2}{3} Ae - Ae - 2c \times Ana$ for the segment of the spindle aAe .

2) 64.

$$\begin{array}{r}
 2)64.34496 = \text{length of the arc } ACB \\
 \hline
 32.17248 \\
 25 = \text{radius } OC.
 \end{array}
 \left. \vphantom{\begin{array}{r} 2)64.34496 \\ 32.17248 \end{array}} \right\} \text{by last problem.}$$

$$\begin{array}{r}
 16086240 \\
 6434496 \\
 \hline
 \end{array}$$

$$804.31200 = \text{area of the sector } BCAe.$$

168

$$2)636.312 = \text{area of the segment } ACBe.$$

$$318.156$$

$$7 = \text{central distance } eo.$$

$$227.092$$

$$24 = \frac{1}{2} \text{ the diameter } AB.$$

$$24$$

$$96$$

$$48$$

$$576$$

$$24$$

$$2304$$

$$1152$$

$$3)13824$$

$$4608 = \frac{1}{2} \text{ of the cube of } \frac{1}{2} AB.$$

$$2327.092$$

$$2380.908$$

$$2380.908$$

$$\begin{array}{r}
 2380.908 \\
 12.5664 \\
 \hline
 9523632 \\
 14285448 \\
 14285448 \\
 11904540 \\
 4761816 \\
 2380908 \\
 \hline
 \end{array}$$

29919.4422912 = *solidity required.*

2. If the length of a circular spindle be 40, and its middle diameter 30: what is its solidity?

Ans. 17312.858.

PROBLEM XIX.

To find the solidity of the middle frustum of a circular spindle, the length of the frustum, the middle diameter, and that of either of the ends being given.

R U L E. *

1. Divide the square of half the length of the frustum by half the difference of the middle diameter, and that of either of the two ends, and half this quotient added to $\frac{1}{2}$ of the said difference will give the radius of the circle.

2. Find the central distance, and the revolving area, as in the last problem.

3. From the square of the radius take the square of the central distance, and the square root of the remainder will give half the length of the spindle.

* The demonstration is contained in that of the last problem.

The solidity of the segment cannot be found independant of the length of the spindle. The rule may be seen in the Corollary to the last problem.

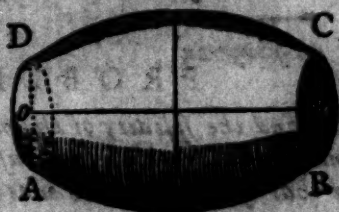
3. From

4. From the square of half the length of the spindle take $\frac{1}{3}$ of the square of half the length of the frustum, and multiply the remainder into the said half length.

5. Take this product from that of the generating area and central distance, and the remainder multiplied by 6.2832 will give the content of the frustum.

EXAMPLES.

1. What is the solidity of the frustum ABCD, whose middle diameter nm is 36, the diameter DA or BC 16, and the length or 40?



$$36 = nm.$$

$$16 = D d.$$

$$2)20$$

$$10 = \frac{1}{2} \text{ diff. and}$$

$$5 = \frac{1}{2} \text{ diff.}$$

$$20$$

$$20$$

$$10)400 = \text{square of } \frac{1}{2} \text{ or}$$

$$2)40$$

$$20$$

$$5$$

$$25 = \text{radius, and } 50 = \text{diameter.}$$

$$18 = \frac{1}{2} nm.$$

$$7 = \text{central distance.}$$

$$50$$

$$50)10.0$$

.2 = tabular versed sine.

.111823 = tabular segment.

2500 = square of the diameter.

$$\begin{array}{r} 55911500 \\ 223646 \end{array}$$

$$279.557500 = \text{area of the segment } D \cap C.$$

$$599.5575 = \text{generating area of } D \cap C.$$

7 = central distance.

$$4196.9025$$

$$\begin{array}{l} 40 = \text{cr.} \\ 8 = D \cap \end{array}$$

$$320 = \text{area of } D \cap C.$$

$$\begin{array}{r} 25 \\ 25 \\ \hline 125 \\ 50 \end{array}$$

625 = square of the radius.

49 = square of the central dist.

$$576 (24 = \frac{1}{2} \text{ the length of the spindle.})$$

$$\begin{array}{r} 4 \\ \hline 44)176 \\ 176 \end{array}$$

24

24

—

96

48

—

576 = square of $\frac{1}{2}$ the length of the spindle.

133.3333 = $\frac{1}{3}$ square of $\frac{1}{2}$ or.

—

442.6667

20 = $\frac{1}{2}$ or.

—

8853.3340

4196.9025

—

4656.4315

6.2832

—

93128630

139692945

372514520

93128630

279385890

29257.29040080 = solidity required.

2. The middle diameter of the frustum of a circular spindle is 32, the diameter at the end 24, and the length 40: what is the solidity?

Ans. 27286.5411256 cubic inches.

PROBLEM XX.

To find the solidity of a spheroid.

RULE.

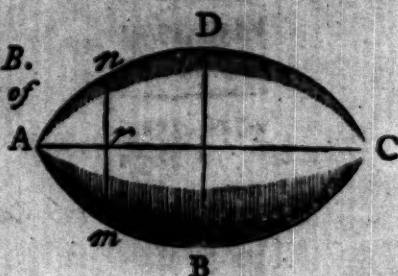
RULE.*

Multiply the square of the revolving axe by the fixed axe, and this product again by .5236, and it will give the solidity required.

EXAMPLES.

1. In the prolate spheroid ABCD, the transverse or fixed axe AC is 90, and the conjugate or revolving axe DB is 70: what is the solidity?

$$\begin{array}{r}
 70 \\
 70 \\
 \hline
 4900 = \text{square of} \\
 \quad \quad \quad DB. \\
 90 = AC. \\
 \hline
 441000 \\
 .5236 \\
 \hline
 2646000 \\
 1323000 \\
 882000 \\
 2205000 \\
 \hline
 230907.6000 = \text{solidity required.}
 \end{array}$$



2. What

* *Demon.* Let $AC = a$, $DB = b$, $Ar = x$, $rn = y$, and $p = 3.14159$ &c.

Then $a^2 : b^2 :: x \times a - x^2 : \frac{b^2}{a^2} \times ax - x^2 (y^2)$ by the property of the ellipsis.

And therefore the fluxion of the solid $(= xy^2 \dot{x}) = \frac{pb^2}{a^2} \times$

$ax\dot{x} - x^2\dot{x}$; and its fluent $= \frac{pb^2}{a^2} \times \frac{1}{2}ax^2 - \frac{1}{3}x^3 = \text{segment}$
 $nAm,$

2. What is the solidity of a prolate spheroid, whose fixed axe is 100, and its revolving axe 60?

Ans. 188496.

3. What is the solidity of an oblate spheroid, whose fixed axe is 60, and its revolving axe 100?

Ans. 314160.

PROBLEM XXI.

To find the content of the middle frustum of a spheroid.

CASE I.

When the ends are circular, or parallel to the revolving axis.

RULE.*

To twice the square of the middle diameter add the square of the diameter of either of the ends, and this sum multiplied by the length of the frustum, and the product again by .2618, will give the solidity.

EX-

nAm. Which, when $x = a$, becomes $\frac{pb^2}{a^2} \times \frac{1}{3}a^3 - \frac{1}{3}a^3 = \frac{pab^2}{6} = \text{content of the whole spheroid. Q. E. D.}$

Let f = fixed axe, r = revolving axe, $q = \frac{f^2 \cos^2 \theta}{f^2}$, and $p = 3.1415$ &c.

Then will $prf\sqrt{1+\frac{1}{3}q}$ = surface of the oblate spheroid, and $prf\sqrt{1-\frac{1}{3}q}$ = that of the prolate spheroid.

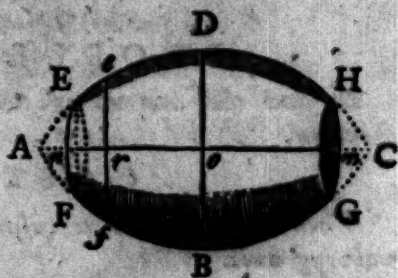
* *Demon.* Let $Ao = a$, $Do = b$, $En = b$, $no = c$, $ro = x$, $re = y$, and $p = 3.14159$ &c.

Then $a : b^2 :: a^2 - x^2 : \frac{b^2}{a^2} \times \frac{b^2}{a^2} - x^2 = b^2 - \frac{b^2 x^2}{a^2} = y^2$
by the property of the ellipsis.

And

EXAMPLES.

1. In the middle frustum of a spheroid EFGH, the middle diameter DB is 50 inches, and that of either of the ends EF or GH 40 inches, and its length n 18 inches: what is its solidity?



$$\begin{array}{r}
 50 \\
 50 \\
 \hline
 2500 \\
 2 \\
 \hline
 5000 = \text{twice the square of } DB \\
 1600 = \text{square of } EF \text{ or } HG \\
 \hline
 6600 \\
 18 = \text{length } n \\
 \hline
 52800 \\
 6600 \\
 \hline
 118800 \\
 .2618 \\
 \hline
 950400 \\
 118800 \\
 712800 \\
 237600 \\
 \hline
 31101.8400 \text{ cubic inches, the answer.}
 \end{array}$$

2. What

And also $a^2 + b^2 :: a^2 - c^2 : \frac{b^2 \times a^2 - c^2}{a^2} = b^2$, or a^2

$$= \frac{b^2 c^2}{b^2 - a^2}.$$

Now,

2. What is the solidity of the middle frustum of a prolate spheroid, the middle diameter being 60, that of either of the two ends 36, and the distance of the ends 80? *Ans.* 177940.224.

3. What is the solidity of the middle frustum of an oblate spheroid, the middle diameter being 100, that of either of the ends 80, and the distance of the ends 36? *Ans.* 248814.72.

CASE II.

When the ends are elliptical, or perpendicular to the revolving axis.

RULE.*

i. Multiply

Now, by substituting this value of a^2 in the former equation, we shall have $y^2 = b^2 - \frac{b^4x^2 - b^2b^2x^2}{b^2c^2} = b^2 -$

$$\frac{b^2x^2 - b^2x^2}{c^2} = b^2 - \frac{x^2 \times \overline{b^2 - b^2}}{c^2}.$$

And consequently the fluxion of the solid $(py^2\dot{x}) = p b^2 \dot{x} - \frac{p x^2 \dot{x} \times \overline{b^2 - b^2}}{c^2}$; the fluent of which is $= p b^2 x - \frac{p x^3 \times \overline{b^2 - b^2}}{3c^2}$,

which, when $x=c$, becomes $p b^2 c - \frac{p c b^2 - p c b^2}{3} = \frac{p c \times \overline{8b^2 + 4b^2}}{12}$. Q. E. D.

* *Demon.* Put $so = a$, $Ev = b$, $om = r$, $on = x$, $An = y$, $ac = z$, and $p = 3.14159$ &c.

Then $a^2 : b^2 :: a^2 - x^2 : \frac{b^2 \times \overline{a^2 - x^2}}{a^2} = y^2$ by the property of the ellipsis.

And, since Acd is an ellipsis similar to EmF , it will be $\phi : r :: y : \frac{ry}{b} = z$.

I

But

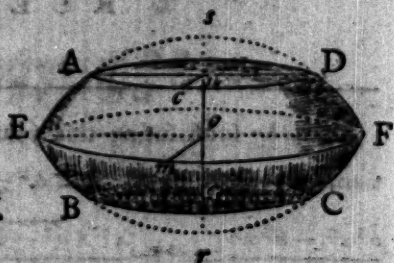
1. Multiply twice the transverse diameter of the middle section by its conjugate diameter, and to this product add the product of the transverse and conjugate diameters of either of the ends.

2. Multiply the sum, thus found, by the distance of the ends, and the product again by .2618, and it will give the solidity required.

EXAMPLES.

1. In the middle frustum ABCD of an oblong or prolate spheroid, the diameters of the middle section EF are 50 and 30; those of the end AD 40 and 24; and its height *sc* 18; what is the solidity?

$$\begin{array}{r}
 50 \\
 \times 2 \\
 \hline
 100 = \text{twice the trans.} \\
 30 = \text{conjugate} \\
 \hline
 3000
 \end{array}$$



40

But the fluxion of the solid AEFD is $py\dot{x} = py\dot{x} \times$

$$\frac{ay}{b} = \frac{p r y^2 \dot{x}}{b} = \frac{p r \dot{x}}{b} \times \frac{b^2 \times a^2 - x^2}{a^2} = p r b \dot{x} \times \frac{a^2 - x^2}{a^2}. \text{ And}$$

the fluent $= p r b x \times \frac{a^2 - \frac{1}{2}x^2}{a^2}$. Which, by substituting for

$$a^2 \text{ its value } \frac{b^2 x^2}{b^2 - y^2}, \text{ becomes } = p r x \times \frac{2b^2 + y^2}{3b} = p r x \times$$

$$\frac{2rb + \frac{ry^2}{b}}{3}.$$

3

and

40 = *transverse*
24 = *conjugate*

160

80

960

3000

960

3960

18

31680

3960

71280

.2618

570240

71280

427680

142560

18661.1040 *solidity required.*

2. In the middle frustum of a prolate spheroid, the diameters of the middle section are 100 and 60; those of the end 80 and 48; and the length 36: what is the solidity? *Ans.* 149288.832.

And this again, by putting x for its equal $\frac{7}{8}$, becomes =

$$\frac{p \times}{3} \times \overline{2rb + y^2} = \text{frustum EFDA. Or } \frac{p \times nc}{12} \times 2EF \times 2om$$

$$+ AD \times 2nc = \text{middle frustum ABCD. } Q. E. D.$$

3. In the middle frustum of an oblate spheroid, the diameters of the middle section are 100 and 60; those of the end 60 and 36; and the length 80: what is the solidity of the frustum?

Ans. 296567.04.

PROBLEM XXII.

To find the solidity of the segment of a spheroid.

CASE I.

When the base is circular, or parallel to the revolving axis.

RULE.*

1. Divide the square of the revolving axis by the square of the fixed axe, and multiply the quotient by the difference between three times the fixed axe and twice the height of the segment.

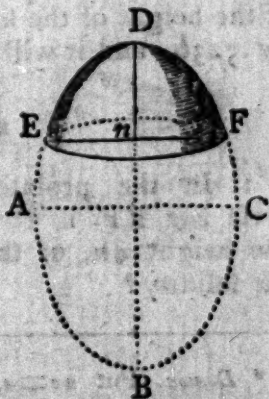
2. Multiply the product, thus found, by the square of the height of the segment, and this product again by .5236, and it will give the solidity required.

EXAMPLES.

1. In the prolate spheroid DABC, the transverse axis DB is 100, the conjugate AC 60, and the height Dn of the segment EDF 10: what is the solidity?

* This rule is formed from the theorem for the segment in the demonstration to problem the 20th.

$$\begin{array}{r}
 60 \\
 60 \\
 \hline
 \text{Square of } DB = 10000 \quad 3600 = \text{Square of } AC \\
 \hline
 .36 \qquad \qquad \qquad 2 D n \\
 280 = \text{diff. between } 3 DB \text{ and} \\
 \hline
 2880 \\
 72 \\
 \hline
 100.80 \\
 \text{Square of } D n = 100 \\
 \hline
 10080 \\
 .5236 \\
 \hline
 60480 \\
 30240 \\
 20160 \\
 50400 \\
 \hline
 5277.8880 = \text{solidity required.}
 \end{array}$$



2. The axis of a prolate spheroid are 50 and 30: what is the solidity of that segment whose height is 5, and its base perpendicular to the fixed axe?

Ans. 659.736.

3. The diameters of an oblate spheroid are 100 and 60; what is the solidity of that segment whose height is 12, and its base perpendicular to the conjugate axe?

Ans. 32672.64.

CASE II.

When the base is elliptical, or perpendicular to the revolving axis.

I 3

RULE.

R U L E.*

1. Divide the fixed axe by the revolving axe, and multiply the quotient by the difference between three times the revolving axe and twice the height of the segment.

2. Multiply the product, thus found, by the square of the height of the segment, and this product again by .5236, and it will give the solidity required.

E X A M P L E S.

1. In the prolate spheroid $aEbF$, the transverse axe EF is 100, the conjugate ab 60, and the height an , of the segment aAD , 12: what is the solidity?

* *Demon.* Put $ao=a$, $EO=b$, $om=r$, $an=x$, $As=y$, $ac=z$, and $p=3.14159$, &c. Then will $a^2:b^2::a^2-(a-x)^2(2ax-x^2):\frac{b^2 \times 2ax-x^2}{a^2}=y^2$ by the property of the ellipse.

And, since AeD is an ellipse similar to EaF , it will be $b:r::y:\frac{xy}{b}=z$.

But the fluxion of the solid $aAeD$ is $=pyz\dot{x}=py\dot{x} \times \frac{xy}{b} = \frac{pry^2\dot{x}}{b} = \frac{pr\dot{x}}{b} \times \frac{b^2 \times 2ax-x^2}{a^2}$; whose fluent is $= \frac{prb}{a}x^2 - \frac{prb}{3a^2}x^3$; which when $\dot{x}=b$ = the height of the segment, becomes $\frac{3ab^2-b^3}{3a^2} \times \frac{prb}{3a^2}$. Whence, since $r=a$, we shall have $\frac{3ab^2-b^3}{3a^2} \times \frac{pb}{3a}$ = solidity of the segment.

Q. E. D.

$156 = \text{diff. between } 3ab \text{ and } 2an$

$1\frac{1}{2} = EF, \text{ or } 100, d. \text{ divided by } ab, \text{ or } 60$

$$\begin{array}{r} 156 \\ 2 = \frac{1}{2} \quad 78 \\ 4 = \frac{1}{3} \quad 26 \end{array}$$

260

$144 = \text{square of } ab$

1040

1040

260

37440

.5236

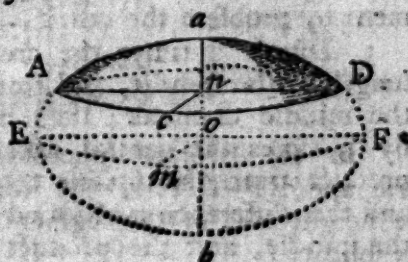
224640

112320

74880

187200

$19603.5840 = \text{solidity required.}$



PROBLEM XXIII.

To find the solidity of an elliptical spindle; the length or fixed axe, the middle diameter, and another parallel thereto at $\frac{1}{4}$ of the length of the spindle, being given.

RULE.*

1. From

* Démon. Let $Fv = a$, ov or $ma = b$, $Ao = c$, $Do = d$, va or $cm = x$, and ve or $an = y$.

1. From three times the square of the middle diameter take 4 times the square of the diameter between the middle and the end; and from 4 times this last diameter take 3 times the said middle diameter, and $\frac{1}{4}$ of the quotient arising from dividing the former difference by the latter will give the *central distance*.

2. Find the axes of the ellipsis by problem the 2d, in conic sections, and the area of the generating segment by problem the 5th.

3. Divide 3 times the area, thus found, by the length of the spindle, and from the quotient subtract the middle diameter; then multiply the remainder by 4 times the central distance, and subtract the product from the square of the middle diameter; and this difference multiplied by $\frac{1}{3}$ of the length of the spindle, and the product again by 1.57079 will give the solidity.

EXAMPLES.

Then, by the property of the ellipse, $c:d::\sqrt{c^2-x^2}$;
 $\frac{d}{c} \sqrt{c^2-x^2} = nm$; hence $an = mn - ma = \frac{d\sqrt{c^2-x^2}}{c} - b = y$, and the fluxion of the solid $= py^2\dot{x} = p\dot{x} \times$
 $d^2 - \frac{d^2x^2}{c} - \frac{2bd\sqrt{c^2-x^2}}{c} + b^2 = p\dot{x} \times d^2 - b^2 - \frac{d^2x^2}{c^2} -$
 $\frac{2bd\sqrt{c^2-x^2}}{c} - b = pd^2\dot{x} \times \frac{d^2-x^2}{c^2} - 2pby\dot{x}$; and its
 fluent $= pd^2x \times \frac{3a^2-x^2}{3c^2} - 2pb \times \text{area } wand = \text{frustum}$
 Dxw . And when $x=a$ the above theorem will become
 $\frac{2pd^2}{3c^2} \times a^3 - 2pb \times \text{area } FDw = DFw$ the half of the
 spindle. And if from the semi-spindle there be taken the
 frustum, there will remain $pd^2c^2 \times \frac{3a-c}{3c^2} - 2pb \times \text{area}$
 Fna

EXAMPLES.

1. What is the solidity of the elliptic spindle $FwGD$, whose length FG is 80, the middle diameter Dw 24, and the diameter rs at $\frac{1}{4}$ of the length 18.99094?

1. For the central distance, and the axes of the ellipsis.

$$4rs = 75.96376$$

$$3Dw = 72.$$

$$3.96376$$

$$1728 = 3 \text{ times the square of } Dw$$

$$1442.62320833 = 4 \text{ times the square of } rs$$

(4)

$$3.96376) 285.37679167 (71.996$$

$$277 \ 4632$$

cent. dist. ow

$$17.999 = 18 \text{ nearly, for the}$$

$$791359$$

$$396376$$

$$3949831$$

$$3567384$$

$$3824476$$

$$3567384$$

$$2570927$$

$$2378256$$

$$192671$$

$$18 + 12 =$$

$$30 = \frac{1}{2} \text{ conjugate.}$$

900

Fna = segment nFx , s being the height Fa of the segment. But to convert these rules into those given in the text, let $Dw = D$, $rx = d$, $rs = m$, $va = b$, $av = C$, the segment $nDe = s$, and $n = .7854$.

15

Then,

$$900 = \text{square of } D w$$

$$324 = \text{square of } o w$$

$$\hline 576(24$$

$$\hline 4$$

$$44) 176$$

$$\hline 176$$

$$30 = D w$$

$$40 = F w$$

$$24) 1200(50 = \frac{1}{2} \text{ transverse diameter } AB$$

$$\hline 120$$

$$\hline 0$$

2. For the elliptic segment FDG.

$$DE = 60) 12 = D w$$

.2 = tabular versed sine.

$$.111823 = \text{circular segment belonging to .2}$$

$$\hline 100 = AB$$

$$\hline 11.182300$$

$$\hline 60 = DE$$

$$670.938000 = \text{generating segment FDG.}$$

Then, by the property of the ellipse, $C + \frac{1}{2}D^2 - C + \frac{1}{2}d^2 : 4 :: C + \frac{1}{2}D^2 - C + \frac{1}{2}d^2 : 1$; hence $4 \times C + \frac{1}{2}d^2 - 3 \times C + \frac{1}{2}D^2 = C + \frac{1}{2}d^2$, or $C = \frac{1}{2} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$.

And the last two theorems become $\frac{1}{2} w b \times 2D^2 + d^2 - 2 \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m} \times -D + d + \frac{3s}{b}$ for the frustum $D s w$;

and $\frac{1}{2} w l \times 2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m} \times -D + \frac{3S}{l}$ for the semi-spindle $DF w$, where $l = 2F$ and $S = \text{area of } FD w$.

Q. E. D.

3. For

3. For the solidity of the spindle.

670.938

3

$FG = 8.0$ 2012.814

25.160175

24 = Dw

1.160175

72 = 4 times ov

2320350

8121225

83.532600

576 = square of Dw

83.5326

492.4674

80 = FG

3)39397.3920

13132.464

1.57079

118192176

91927248

919272480

65662320

13132464

20628.34312656 = solidity required.

2. The length of an elliptic spindle is 40, the middle diameter 12, and the diameter at $\frac{1}{4}$ of the length 9.49546: what is the solidity? *Ans.* 2578.56.

PROBLEM XXIV.

To find the solidity of the middle frustum of an elliptic spindle; the length, the diameters of the middle and end, and another parallel thereto at $\frac{1}{4}$ of the length of the frustum, being given.

RULE.*

1. From the sum of 3 times the square of the middle diameter and the square of that of the end, take 4 times the square of the diameter between the middle and the end; and from 4 times this last diameter take the sum of the least diameter and 3 times that of the middle, and $\frac{1}{4}$ of the quotient arising from dividing the former difference by the latter will give the *central distance*.

2. Find the axes of the ellipse by problem the 2d, and the area of the elliptical segment whose chord is the length of the frustum by problem the 5th.

3. Divide the area thus found by the length of the frustum, and from the quotient subtract the difference between the middle diameter and that of the end, and multiply the remainder by 8 times the central distance; then from the sum of the square of the least diameter, and twice the square of that in the middle, take the product last found, and this difference multiplied by the length, and the product again by .261799, &c. will give the solidity required.

* The demonstration of this rule is contained in that of the last problem.

The solidity of a segment of an elliptic spindle cannot be found independently of the axes of the spheroid.

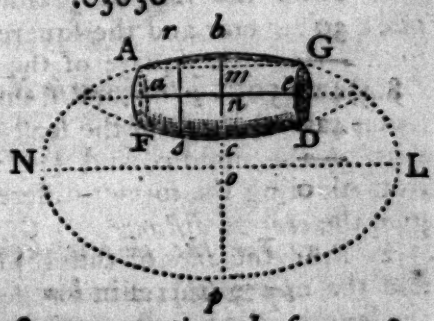
EXAMPLES.

EXAMPLES.

1. What is the solidity of the frustum AFDC, whose length ac is 28, the middle diameter bc 24, the diameter AF of the end 21.6, and that rs at $\frac{1}{4}$ of the length 23.40909?

1. For the central distance, and the axes of the ellipse.

24 = bc	93.63636 = 4 times rs
24	93.6 = sum of AF and 3 times bc
96	.03636
48	
576	
3	
1728	
$\square$ of AF = 466.56	
2194.56	
2191.9419785124 = 4 times the square of rs .	
	(4)
.03636) 2.6180214876 (72.00279	
2 5452	
7282	18.00092 = 18 nearly =
7272	central dist. on.
10148	
7272	
28767	
25452	
33156	
32724	
432	



$$900 = \text{square of } bc$$

$$829.44 = \text{square of } om$$

$$70.56(8.4$$

$$64$$

$$164)656$$

$$656$$

$$30 = bc$$

$$14 = an$$

$$120$$

$$30$$

$$8.4)420(50 = \frac{1}{2} \text{ transverse } No, \text{ and } 100 = NL$$

$$420$$

$$0$$

2. For the elliptic segment $AbGm$,

$$bp = 60) 1.2 = bm$$

$$.02 = \text{tabular versed sine.}$$

$$.003748 = \text{circular segment belonging to } .02.$$

$$100 = NL$$

$$.374800$$

$$60 = bp$$

$$22.488000 = \text{elliptic segment } AbGm.$$

3. For the solidity of the frustum.

$$22.4880$$

$$3$$

$$28)67.4640(2.4094$$

$$56$$

$$2.4 = \text{diff. of } bc \text{ and } AF$$

$$114 \&c.$$

$$.0094$$

$$.0094$$

$$\begin{array}{r} 111.0094 = \text{diff. of } bc \text{ and } AF \\ 144 \\ \hline 376 \\ 376 \\ \hline 94 \end{array}$$

$$1.3536$$

$$\begin{array}{r} 24 \\ 24 \\ \hline 96 \\ 48 \end{array}$$

$$576$$

$$1152 = \text{twice the square of } bc$$

$$466.56 = \text{square of } AF$$

$$1618.56$$

$$1.3536$$

$$1617.2064$$

$$28 = ac$$

$$129376512$$

$$32344128$$

$$45281.7792$$

$$.261799$$

$$4075360128$$

$$4075360128$$

$$3169724544$$

$$452817792$$

$$2716906752$$

$$905635584$$

$$11854.7245127808 = \text{solidity required.}$$

2. In the middle frustum of an elliptic spindle, the middle diameter is 32, the diameter at the end 24, and the diameter at $\frac{1}{4}$ of the length 30.15756, and the length 40: required the solidity.

Ans. 27419.8219.

PROBLEM XXV.

To find the solidity of a parabolic conoid.

RULE.*

Multiply the area of the base by half the altitude, and the product will be the content.

EXAMPLES.

1. What is the solidity of the paraboloid B A D, whose height D m is 84, and the diameter B A of its circular base 48?

* *Demon.* Let Dm = a , Bm = b , Dn = x , En = y , and $p = 3.14159$, &c.

Then by the nature of the parabola $a : b^2 :: x : y^2$, or $\frac{b^2 x}{a} = y^2$; wherefore $\frac{p b^2 x \dot{x}}{a}$ (= $p y^2 \dot{x}$) = the fluxion

of the solid, and $\frac{p b^2 x^2}{2a}$ = its fluent; which, when x becomes = a , is $\frac{1}{2} p a b^2$ for the whole solid, or for any segment whose height is = a , and the radius of its base = b . Q. E. D.

Coroll. The parabolic conoid is = $\frac{1}{2}$ its circumscribing cylinder.

Note. The rule given above will hold for any segment of the paraboloid, whether the base be perpendicular or oblique to the axe of the solid.

$$\begin{array}{r}
 48 \\
 48 \\
 \hline
 384 \\
 192 \\
 \hline
 \end{array}$$

2304 = square of BA

$$\begin{array}{r}
 7854 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 9216 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 11520 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 18432 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 16128 \\
 \hline
 \end{array}$$

1809.5616 = area of the base

$$42 = \frac{1}{2} Dm$$

$$\begin{array}{r}
 36191232 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 72382464 \\
 \hline
 \end{array}$$

76001.5872 = solidity required.

2. What is the solidity of a paraboloid, whose height is 60, and the diameter of its circular base 100?

Ans. 235620.

3. Required the solidity of a parabolic conoid, whose height is 30, and the diameter of its base 40?

Ans. 18849.6.

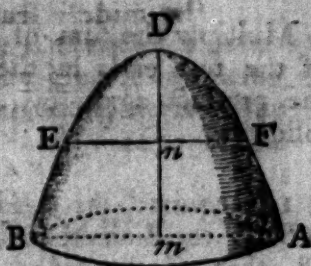
PROBLEM XXVI.

To find the solidity of the frustum of a paraboloid, when its ends are perpendicular to the axe of the solid.

RULE.

Multiply

* *Demon.* The segment whose base is B, and altitude A, is $= \frac{1}{2} AB$; and that whose base is b and altitude a is $= \frac{1}{2} ab$, by

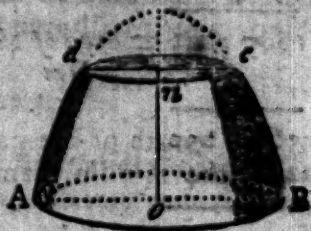


Multiply the sum of the squares of the diameters of the two ends by the height of the frustum, and the product again by .3927, and it will give the solidity.

EXAMPLES.

1. Required the solidity of the parabolic frustum $ABcd$, the diameter AB of the greater end being 58, that of the lesser end dc 30, and the height no 18.

$$\begin{array}{r}
 58 \\
 58 \\
 \hline
 464 \\
 290 \\
 \hline
 \square \text{ of } AB = 3364 \\
 \square \text{ of } dc = 900 \\
 \hline
 4264 \\
 18 = \text{height no} \\
 \hline
 34112 \\
 4264 \\
 \hline
 76752 \\
 .3927 \\
 \hline
 537264 \\
 153504 \\
 690768 \\
 230256 \\
 \hline
 30140.5104 = \text{solidity required.}
 \end{array}$$



by the last problem: wherefore the frustum, or the difference of the segments is $\frac{1}{2}AB - \frac{1}{2}ab$. But $B-b : A-a (d) :: B : A = \frac{Bd}{B-b}$; and $B-b : d :: b : a = \frac{bd}{B-b}$, by the nature of the paraboloid; and these values of A and a being substituted for them will make $\frac{1}{2}AB - \frac{1}{2}ab = \frac{dB - db}{2B - 2b} = \frac{1}{2}d \times B + b$, which is the same as the rule. Q. E. D.

2. What

2. What is the solidity of the frustum of a parabolic conoid, the diameter of the greater end being 60, that of the lesser end 48, and the distance of the ends 18?

Ans. 41733.0144

PROBLEM XXVII.

To find the solidity of a parabolic spindle.

R U L E.*

Multiply the square of the middle diameter by the length of the spindle, and the product again by .418879, and it will give the solidity.

EXAMPLES.

1. The length of the parabolic spindle ACBD is 60, and the middle diameter DC 34: what is the solidity?

* *Demon.* Put $Dc = a$, $Ac = b$, $c = 3.14159$, &c. $An = x$, and $rn = y$.

Then it will be, by the property of the parabola, $b^2 : a :: 2bx - x^2 (An \times nB) : a \times \frac{2bx - x^2}{b^2} = rn = y$; which, by putting $p = \frac{b^2}{a}$ (= parameter of Dc), becomes $\frac{2bx - x^2}{p}$;

and hence the fluxion of the solid $= cy^2 \dot{x} = \frac{cx}{p^2} \times \frac{2bx - x^2}{2} \dot{x}$;

the fluent of which is $cx \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{5}x^2}{p^2} =$ general ex-

pression for the segment ras ; and therefore when $x = b$, we shall have $\frac{8cb^5}{15p^2} = \frac{8}{15}ca^2b =$ semi-spindle DAC, and $a^2b \times .418879 =$ whole spindle BDAC. Q. E. D.

34

34

136

102

1156 = square of DC
60 = AB

69360

.418879

624240

485520

554880

554880

69360

277440

29053.447440 = solidity required.

2. The length of a parabolic spindle is 9 feet, and the middle diameter 3 feet: what is the solidity?

Ans. 33.929199.

PROBLEM XXVIII.

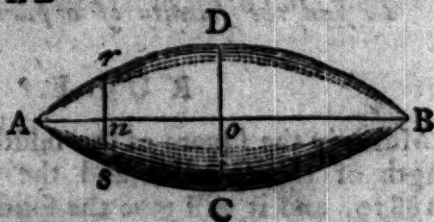
To find the solidity of the middle frustum of a parabolic spindle.

RULE.

Add

* *Demon.* Let $on = a - x = x$, and the other letters as in the last problem.

Then if from the value of the semi-spindle $\frac{8cbs}{15p^2}$ there be taken $cx^3 \times \frac{\frac{4}{3}b^2 - bx + \frac{1}{3}x^2}{p^2}$ the value of the segment EAF, there

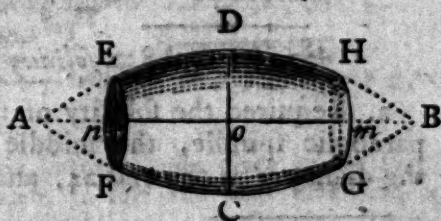


Add 8 times the square of the middle diameter, 3 times the square of the less, and 4 times the product of those diameters into one sum; then this sum being multiplied by the length, and the product again by .05236 will give the solidity.

EXAMPLES.

1. In the middle frustum EFGH, of the parabolic spindle ACBD, the middle diameter DC is 36, the diameter of the end EF is 20, and the length nm 36: what is the solidity?

$$\begin{array}{r}
 36 \\
 36 \\
 \hline
 216 \\
 108 \\
 \hline
 1296 \\
 8 \\
 \hline
 \end{array}$$



$$\begin{array}{l}
 10368 = 8 \text{ times the square of } DC \\
 1200 = 3 \text{ times the square of } EF \\
 2880 = 4 \text{ times the product of } DC \text{ and } EF \\
 \hline
 14448
 \end{array}$$

there will remain $\frac{cx}{p^2} \times \overline{b^4 - \frac{2}{3}b^2x^2 + \frac{1}{3}x^4} = \frac{ca^2x}{b^4} \times \overline{b^4 - \frac{2}{3}b^2x^2 + \frac{1}{3}x^4}$ = the value of the frustum DEFC; and if instead of x there be substituted its value $b\sqrt{\frac{a-y}{a}}$ the value of the said frustum will be denoted by $c \times no \times \frac{8DC^2 + 4no \times EF + 3EF^2}{60}$, and consequently $nm \times .05236 \times 8DC^2 + 4no \times EF + 3EF^2$ = whole frustum. Q. E. D.

14448

$$\begin{array}{r}
 14448 \\
 36 = \text{length } \times m \\
 \hline
 86688 \\
 43344 \\
 \hline
 520128 \\
 .05236 \\
 \hline
 3120768 \\
 1560384 \\
 1040256 \\
 2600640 \\
 \hline
 27233.90208 = \text{solidity required.}
 \end{array}$$

2. Required the solidity of the middle frustum of parabolic spindle, the middle diameter being 32, the diameter at the end 24, and the length 40?

Ans. 96.4907.

PROBLEM XXIX.

To find the solidity of an hyperboloid.

RULE.

* *Demon.* Let t = transverse, and c = conjugate diameter of the generating hyperbola, $p = 3.14159$, y , Y , the ordinates, or semi-diameters of the ends of any frustum of the hyperboloid, x = its altitude, and A = distance of the less ordinate y from the vertex of the whole solid.

Then since $Y^2 = \frac{t^2 + A^2 + 2Ax}{c^2} \times c^2$, we shall have the fluxion of the solid $= pY^2 \dot{x} = pc^2 \dot{x} \times \frac{At + A^2 + 2Ax + \frac{2x + x^2}{x^2}}{c^2}$, and its fluent $= pc^2 x \times \frac{At + A^2 + Ax + \frac{1}{2}x + \frac{1}{2}x^2}{c^2}$; and

To the square of the radius of the base add the square of the middle diameter between the base and the vertex; and this sum multiplied by the altitude, and the product again by .5236 will give the solidity.

EXAMPLES.

1. In the hyperboloid ACB, the altitude CD is 10, the radius AD of the base 12; and the middle diameter *nm* 15.8745: what is the solidity?

and this, by substituting $\frac{y^2}{c^2}$ for $\frac{Ar + A^2}{r^2}$, and $\frac{Y^2}{c^2}$ for $\frac{Ar + A^2 + 2Ax + x^2}{r^2}$ becomes $Y + y^2 - \frac{c^2 x^2}{3r^2} \times \frac{1}{2} p x =$
solidity of the frustum.

But to convert this into the rules given in the text, let *D*, *δ*, *d*, be the greatest, middle and least diameters, *x* = abscissa whose ordinate is *δ*, and *a* = altitude. Then we shall have these three equations:

$$r^2 \delta^2 = c^2 \times r + x \times x$$

$$r^2 d^2 = c^2 \times r + x - \frac{1}{2} a \times x - \frac{1}{2} a^2$$

$$r^2 D^2 = c^2 \times r + x + \frac{1}{2} a \times x + \frac{1}{2} a^2$$

From the sum of the two latter of which subtract the double of the former, and there will result $r^2 \times D^2 - 2\delta^2 + d^2 = \frac{1}{2} a^2 c^2$; and hence $\frac{a^2 c^2}{3r^2} + \frac{2D^2 - 4\delta^2 + 2d^2}{3}$. Which being substituted for it in the theorem above will give $\frac{D^2 + 4\delta^2 + d^2}{6} \times ap$ for the content of the frustum; which is the same as the following rule given in the text.

And if *d* the least diameter be supposed to become infinitely little, or nothing, the rule will become $\frac{D^2 + 4\delta^2}{6} \times ap = D^2 + 4\delta^2 \times a \times .5236$. Q. E. D.

$$\begin{array}{r} 251.9975 = \text{square of } nm \\ 144 = \text{square of } AD \end{array}$$

$$\begin{array}{r} 395.99975 \\ 10 = \text{height } cr \end{array}$$

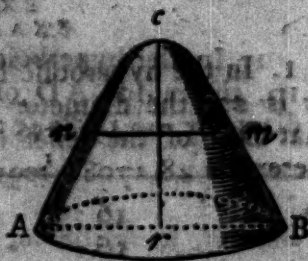
$$\begin{array}{r} 3959.99750 \\ .5236 \end{array}$$

$$2375999850$$

$$1187999925$$

$$791999950$$

$$1979999875$$



$$2073.455869100 = \text{solidity required.}$$

2. In an hyperboloid the altitude is 50, the radius of the base 52, and the middle diameter 68; what is the solidity? *Ans.* 191847.

PROBLEM XXX.

To find the solidity of the frustum of an hyperbolic conoid.

RULE.

Add

* The demonstration of this rule is contained in that of the last problem.

The rule for the hyperbolic spindle is the same as that for the elliptic spindle, page 175.

Or, if D = middle diameter, m = that at $\frac{1}{2}$ of the length, S = generating area of the hyperbola, L = length of the spindle, and $p = 3.14159$, &c.

$$\text{Then will } D + \frac{4m^2 - 3D^2}{4m - 3D} \times \frac{3S}{L} - D \times \frac{1}{2}pL = \text{solidity}$$

of

Add together the squares of the greatest and least semi-diameters, and the square of the whole diameter in the middle, then this sum being multiplied by the altitude, and the product again by .52359 will give the solidity.

EXAMPLES.

1. In the hyperbolic frustum ADCB, the length rs is 20, the diameter AB of the greater end 32, that DC of the lesser end 24, and the middle diameter nm 28.1708: required the solidity.

$$\begin{array}{r}
 16 \\
 16 \\
 \hline
 96 \\
 16 \\
 \hline
 256 = \text{square of } \frac{1}{2} AB \\
 144 = \text{square of } \frac{1}{2} DC \\
 793.5939 = \text{square of } nm \\
 \hline
 1193.5939 \\
 \quad 20 = \text{length } rs \\
 \hline
 23871.878
 \end{array}$$

of the spindle. And if the generating hyperbola be equilateral, then will $3S \times \frac{L^2 + D^2}{LD} - L^2 \times \frac{1}{8} pL = \text{solidity of the spindle.}$

And, if $l = \text{length of the frustum, } s = \text{generating area,}$ and the other letters as before; then will $sD^2 + d^2 + 2 \times \frac{4m^2 - 3D^2 - d^2}{4m - 3d - d} \times \frac{3s}{l} + d - D \times \frac{1}{12} pl = \text{solidity of the middle frustum of an hyperbolic spindle.}$

But if the generating hyperbola be equilateral, the frustum will be $= \frac{3}{2} d^2 - \frac{1}{2} L^2 + \frac{3s}{l} \times \frac{L^2 + D^2 - d^2}{D - d} \times \frac{1}{6} pl.$

K

23871.878

23871.878

.52359

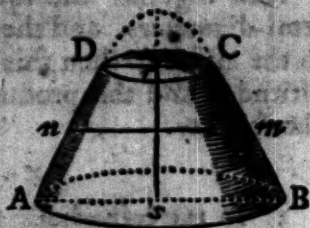
214846902

119359390

71615634

47743756

119359390



12499.07660202 = *solidity required.*

2. What is the solidity of the frustum of an hyperbolic conoid, whose greater diameter is 96, lesser diameter 54, the middle diameter 76.4264392, and the altitude 12.5?

Ans. 116160.66.

OF THE REGULAR BODIES.

A *Regular body* is a solid contained under a certain number of similar and equal plain figures.

The whole number of regular bodies which can possibly be formed is five.

1. The *Tetraedron*, or regular pyramid, which has four triangular faces.

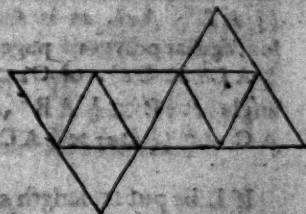
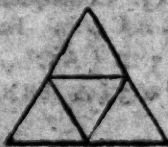
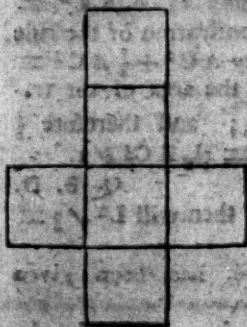
2. The *Hexaedron*, or cube, which has six square faces.

3. The *Octaedron*, which has eight triangular faces.

4. The *Dodecaedron*, which has twelve pentagonal faces.

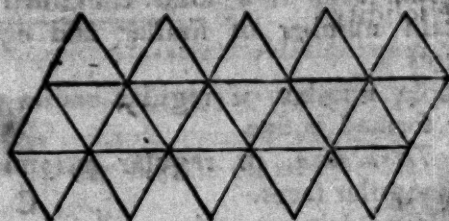
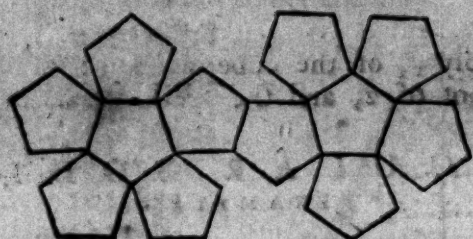
5. The *Icosaedron*, which has twenty triangular faces.

If the following figures are made of pasteboard, and the lines be cut half through, so that the parts may be turned up and glued together, they will represent the five regular bodies here mentioned.



K 2

P R O.



PROBLEM I.

To find the solidity of a tetraedron.

RULE.

* *Demon.* From one angle C of the tetraedron ABC , let fall the perpendicular Ce , upon the opposite side, and draw Ae .

Then $AC^2 - Ae^2 = Ce^2$; and since the point e is equally distant from the three angles A, B and π , $\frac{1}{3} AC^2 = (\frac{1}{3} AB^2) Ae^2$, as is shewn in the demonstration of the rule for regular polygons page 53. Consequently $AC^2 + \frac{1}{3} AC^2 = \frac{4}{3} AC^2 = Ce^2$, or $Ce = AC \sqrt{\frac{2}{3}}$. But the area of the triangle $A\pi B = \frac{1}{2} AB^2 \sqrt{3} = \frac{1}{2} AC^2 \sqrt{3}$; and therefore $\frac{1}{2} AC \sqrt{\frac{2}{3}} (\frac{1}{2} C\pi) \times \frac{1}{2} AC^2 \sqrt{3} (\Delta A\pi B) = \frac{1}{6} AC^3 \sqrt{2}$.

Q. E. D.

If L be put = length of the linear edge, then will $L^3 \sqrt{3} =$ whole surface of the tetraedron.

The rule for the hexaedron, or cube, has been given before.

Multiply

Multiply $\frac{1}{12}$ of the cube of the linear side by the square root of 2, and the product will be the solidity.

EXAMPLES.

1. The linear side of the tetraedron ABC is 4: what is the solidity?

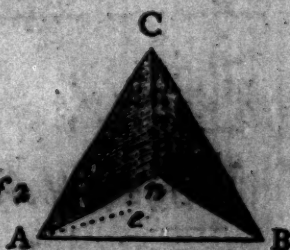
$$\begin{array}{r} 4 \\ 4 \\ \hline 16 \\ 4 \\ \hline 12)64 = \text{cube of } 4 \end{array}$$

$$\begin{array}{r} 5.3333 \\ 1.414 = \text{square root of } 2 \\ \hline 213332 \\ 53333 \\ 213332 \\ 53333 \end{array}$$

7.5412862 = solidity required.

2. Required the solidity of a tetraedron whose side is 6?

Ans. 25.452.



PROBLEM II.

To find the solidity of an octaedron.

RULE.

Multiply

* Demon. From the angle D of the octaedron DFGA let fall the perpendicular De.

K 3

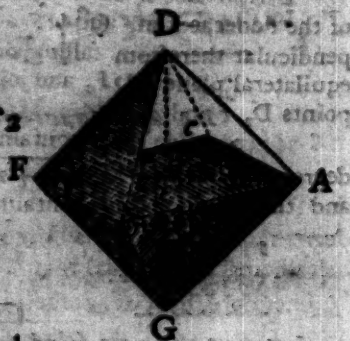
Then,

Multiply $\frac{1}{3}$ of the cube of the linear side by the square root of 2, and the product will be the solidity.

EXAMPLES.

1. What is the solidity of the octaedron FGAD, whose linear side is 4?

$$\begin{array}{r}
 4 \\
 4 \\
 \hline
 16 \\
 4 \\
 \hline
 3)64 = \text{cube of } 4 \\
 \hline
 21.333 \\
 1.414 = \text{square root of } 2 \\
 \hline
 85332 \\
 21333 \\
 85532 \\
 21333 \\
 \hline
 30.164862 = \text{solidity required.}
 \end{array}$$



Then, since the solid is composed of two equal square pyramids, each of whose bases $F \cap A c$ are equal to the square of the linear side AG or AD , we shall have $F \cap A c \times \frac{2}{3} Dc = A n^2 \times \frac{2}{3} Dc = \text{content of the solid.}$

But Dc evidently bisects the diagonal FA , and is equal to Fc , and therefore $A n^2 \times \frac{2}{3} Dc = A n^2 \times \frac{2}{3} Fc = A n^2 \times \frac{1}{3} FA = \frac{1}{3} A n^2 \times \sqrt{A n^2 + A c^2} = \frac{1}{3} A n^2 \sqrt{2} A n^2 = \frac{1}{3} A n^3 \sqrt{2}$. Q. E. D.

If L = linear side as before, then will $2L^3 \sqrt{3} = \text{surface of the octaedron.}$

2. Required

2. Required the solidity of an octaedron whose side is 9? *Ans.* 241.321724.

PROBLEM III.

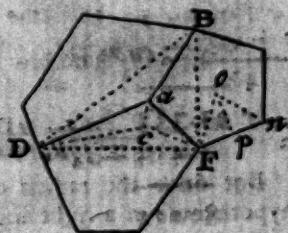
To find the solidity of a dodecaedron.

RULE.*

To

* *Demon.* Let a be a solid angle of the dodecaedron, and ac a perpendicular therefrom falling on the equilateral plane BDF , and join the points D , c and c , F .

Then the angle DcF contains 108 degrees, whose sine is $\frac{1}{2}\sqrt{10+2\sqrt{5}}$, and the angle aFD contains 36 degrees, whose sine is $\frac{1}{2}\sqrt{10-2\sqrt{5}}$, the radius in both cases being taken equal to 1.



Therefore, by Trigonometry, $\frac{1}{2}\sqrt{10-2\sqrt{5}} : \frac{1}{2}\sqrt{10+2\sqrt{5}}$

$$:: aD : DF = aD \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = aD \frac{1+\sqrt{5}}{2}$$

Again, since c is the center of DBF , the angles aDF and cFD are each 30° , and the angle $DcF = 120^\circ$; but the sine of 30° is $\frac{1}{2}$; and the sine of 120° is $\frac{1}{2}\sqrt{3}$; and therefore $\frac{1}{2}\sqrt{3} : DE :: \frac{1}{2} : Dc = \frac{DF}{\sqrt{3}} = AD \frac{1+\sqrt{5}}{2\sqrt{3}}$; and conse-

$$\text{quently } AC = AD^2 - DC^2 = AD \sqrt{\frac{3-\sqrt{5}}{6}}$$

Now a perpendicular from a upon the plane BDF must pass through the center of the circumscribing sphere, and ac will be the versed sine of an arc whose chord is aD , and radius equal to that of the said sphere.

To 21 times the square root of 5 add 47, and divide the sum by 40; then the square root of the quotient being multiplied by 5 times the cube of the linear side will give the solidity required.

$$\text{Whence } ac : aD :: aD : \frac{aD^2}{ac} = \frac{aD^2}{aD\sqrt{3-\sqrt{5}}} = aD$$

$$\frac{\sqrt{3+\sqrt{15}}}{2} = \text{diameter of the circumscribing sphere, and } aD$$

$$\frac{\sqrt{3+\sqrt{15}}}{4} = R = \text{radius of the circumscribing sphere.}$$

Again the angle Fon contains 72° , whose sine is $\frac{1}{2}\sqrt{10+2\sqrt{5}}$; and the angle oFn is 54° , whose sine is $\frac{1+\sqrt{5}}{4}$; and therefore $\frac{1}{4}\sqrt{10+2\sqrt{5}} : \frac{1+\sqrt{5}}{4} :: Fn (aD)$

$$: oF = \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} aD = aD \sqrt{\frac{5+\sqrt{5}}{10}}$$

But since the radius of the circumscribing sphere is the hypotenuse of a right angled triangle, whose two legs are oF , and the radius of the inscribed sphere, we shall have

$$\sqrt{R^2 - oF^2} = \sqrt{\frac{(\sqrt{3+\sqrt{15}})^2}{4} aD^2 - \frac{5+\sqrt{5}}{10} aD^2} = aD$$

$$\sqrt{\frac{25+11\sqrt{5}}{40}} = r = \text{radius of the inscribed sphere.}$$

And because the solid is composed of 12 equal pentagonal pyramids, each of whose bases are by problem the 8th, =

$$\frac{5aD^2}{4} \sqrt{1+\frac{2}{3}\sqrt{5}}; \text{ therefore } \frac{60aD^2}{4} \sqrt{1+\frac{2}{3}\sqrt{5}} \times \frac{1}{3}r =$$

$$\frac{60aD^2}{4} \sqrt{1+\frac{2}{3}\sqrt{5}} \times \frac{aD}{3} \sqrt{\frac{25+11\sqrt{5}}{40}} = 5aD^3 \sqrt{\frac{47+\sqrt{5}}{40}}$$

$$\frac{21\sqrt{5}}{40} = \text{solidity of the dodecaedron. Q. E. D.}$$

$$\text{If } L \text{ be put for the linear side, then will } 15L^2 \sqrt{\frac{5+2\sqrt{5}}{5}} \\ = \text{surface of the dodecaedron.}$$

EXAM-

EXAMPLES.

1. The linear side of the dodecaedron ABCDE is 3: what is the solidity?

$$5(2.23606$$

$$4$$

$$42)100$$

$$84$$

$$443)1600$$

$$1329$$

$$4466)27100$$

$$26796$$

$$447206)3040000$$

$$2683263$$

$$2.23606$$

$$21$$

$$223506$$

$$447212$$

$$46.95726$$

$$47$$

$$49)93.95726$$

$$2.348931(1.5326$$

$$1$$

$$135 = 5 \text{ times the cube of } 3$$

$$25)134$$

$$76830$$

$$125$$

$$45978$$

$$15326$$

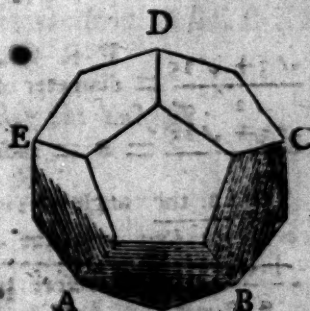
$$303)989$$

$$909$$

$$206.9210 = \text{solidity required.}$$

$$K 5$$

$$3062$$



$$\begin{array}{r}
 3062)8031 \\
 \underline{6124} \\
 30646)190700 \\
 \underline{183876} \\
 6824
 \end{array}$$

PROBLEM IV.

To find the solidity of an icosaedron.

RULE.

To

* *Demon.* Let A be a solid angle of the icosaedron, formed by 5 faces, or triangles, whose bases make the pentagon BCDE.

Then, if a perpendicular be demitted from A upon the pentagonal plane BCDE, it will fall in the center n , and B n , by the demonstration of the

last problem, will be $= AB \sqrt{\frac{5+\sqrt{5}}{10}}$,

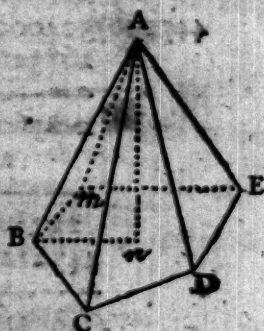
and the radius of the circle circumscribing one of the faces ABC $= \frac{1}{2} AB \sqrt{3}$.

Now the radius of the circumscribing sphere is $R = \frac{BA^2}{2An} = \frac{BA^2}{2\sqrt{AB^2 - Bn^2}} = AB \sqrt{\frac{5+\sqrt{5}}{8}}$, found as in the last problem.

And, since R is the hypotenuse of a right angled triangle, one of whose legs is $\frac{1}{2} AB \sqrt{3}$, the radius of the circle circumscribing the face ABC, and the other r , the radius of the

inscribed sphere, we shall have $r = \sqrt{R^2 - \frac{AB^2}{4} \cdot \frac{3}{4}} = \sqrt{\frac{5+\sqrt{5}}{8} AB^2 - \frac{1}{4} AB^2} = AB \sqrt{\frac{7+3\sqrt{5}}{24}}$.

But

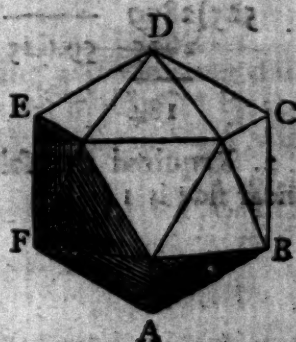


To 3 times the square root of 5 add 7, and divide the sum by 2; then the square root of this quotient being multiplied by $\frac{1}{2}$ of the cube of the linear side will give the solidity required.

EXAMPLES:

1. The linear side of the icosaedron ABCDEF is 3: what is the solidity?

$$\begin{array}{r}
 5(2.23606 \\
 4 \\
 \hline
 42)100 \\
 84 \\
 \hline
 443)1600 \\
 1329 \\
 \hline
 4466)27100 \\
 26796 \\
 \hline
 447206)3040000 \\
 2683236 \\
 \hline
 356764
 \end{array}$$



But the solid is composed of 20 equal triangular pyramids, each of whose bases are $= \frac{AB^2}{4} \sqrt{3}$ by problem 8; there-

fore $\frac{20AB^2}{4} \sqrt{3} \times \frac{1}{2}r = 5AB^2 \sqrt{3} \times \frac{AB}{3} \sqrt{\frac{7+3\sqrt{5}}{24}} = \frac{5}{6} AB^3 \sqrt{\frac{7+3\sqrt{5}}{2}} =$ solidity of the icosaedron. Q. E. D.

If L be put for the linear side, then will $5L^3 \sqrt{3} =$ surface of the icosaedron.

2.3606

3

7.0818

7

2)14.0818

7.0409

4

46)304

276

525)2809

2625

184

(2.65

2 2.5 = $\frac{2}{3}$ cube of 3

1325

530

530

59.625 = *solidity required.*

2. Required the solidity of an icosaedron, whose linear side is 1?

Ans. 2.1816949905.

OF

OF
CYLINDRIC RINGS.

PROBLEM I.

To find the convex superficies of a cylindric ring.

RULE.

TO the thickness of the ring add the inner diameter, and this sum being multiplied by the thickness, and the product again by 9.8696 will give the superficies required.

EXAMPLES.

1. The thickness Ac of a cylindric ring is 3 inches, and the inner diameter cd 12 inches: What is the convex superficies?

* A solid ring of this kind is only a bent cylinder, and therefore the rules for obtaining its superficies, or solidity, are the same as those already given. For let Ac be any section of the solid perpendicular to its axis on , and then $Ac \times 3.14159$, &c. = circumference of that section, and $Ac + cd$ (on) $\times$ 3.14159, &c. = length of the axis on .

Whence $Ac \times 3.14159$, &c. $\times$ $Ac + cd \times 3.14159$, &c. = $Ac + cd \times Ac \times 3.14159$, &c.² = $Ac + cd \times Ac \times 9.8696$, &c. = superficies required.

$$12 = cd$$

$$3 = Ac$$

$$15$$

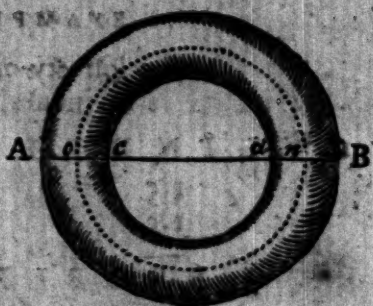
$$3 = Ac$$

$$45$$

$$9.8696$$

$$493480$$

$$394784$$



444.1320 = superficies required.

2. The thickness of a cylindric ring is 4 inches, and the inner diameter 18: what is the convex superficies? *Ans.* 789.57.

PROBLEM II.

To find the solidity of a cylindric ring.

R U L E. *

To the thickness of the ring add the inner diameter, and this sum being multiplied by the square of half the thickness, and the product again by 9.8696 will give the solidity.

$$* \text{ Demon. } Ac^2 \times .78539, \&c. = Ac^2 \times \frac{3.14159}{4}, \&c.$$

$$= \frac{1}{4} Ac^2 \times 3.14159, \&c. = \text{area of the section } Ac; \text{ and}$$

$$Ac + cd \text{ (on)} \times 3.14159, \&c. = \text{length of the axis on.}$$

$$\text{Therefore } \overbrace{Ac + cd} \times \overbrace{\frac{1}{4} Ac^2 \times 3.14159, \&c.}^2 = \overbrace{Ac + cd} \times \frac{1}{4} Ac^2 \times 9.8698. \text{ Q. E. D.}$$

CYLINDRICAL RINGS.

207

EXAMPLES.

1. What is the solidity of an anchor ring, whose inner diameter is 8 inches, and thickness in metal 3 inches?

$$\begin{array}{r}
 1.5 \\
 1.5 \\
 \hline
 75 \\
 15 \\
 \hline
 2.25 = \text{square of } \frac{1}{2} \text{ thickness} \\
 11 = 8 + 3 \\
 \hline
 24.75 \\
 9.8696 \\
 \hline
 14850 \\
 22275 \\
 14850 \\
 19800 \\
 22275 \\
 \hline
 244.272600 = \text{solidity required.}
 \end{array}$$

2. The inner diameter of a cylindric ring is 18 inches, and its thickness 4 inches: what is the solidity? *Ans.* 868.5248.

OF

Take 1/2 of the cube of the diameter, and the sum will be the weight required.

OF THE
WEIGHT AND DIMENSIONS
OF

BALLS AND SHELLS.

THE weight and dimensions of any ball or shell being had from experiment, the weight and dimensions of any other ball or shell, of the same kind, may be found by the following rules.

PROBLEM I.

Having the diameter of an iron shot given, to find its weight.

RULE.

* *Demon.* The weight of an iron shot whose diameter is 4 inches is 9 lbs; and as globes are as the cubes of their diameters, therefore $4^3 : 9 \text{ lbs.} :: D^3$ (diameter being = D) : $\frac{9}{64} D^3$ = weight. But $\frac{9}{64} = \frac{8}{64} + \frac{1}{64} = \frac{1}{8} + \frac{1}{64}$ of $\frac{1}{8}$. Wherefore the weight required is $\frac{1}{8} D^3 + \frac{1}{64}$ of $\frac{1}{8} D^3$.

Q. E. D.

Rule by Logarithms. To 3 times the Log. of the diameter, add — 1.1480625, and the sum is the logarithm of the weight required.

RULE

Take

Take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that eighth, and the sum will be the weight required in pounds, *exactly*.

EXAMPLES:

1. The diameter of an iron shot is 3.5; what is its weight?

$$\begin{array}{r}
 3.5 \\
 3.5 \\
 \hline
 175 \\
 105 \\
 \hline
 18.25 \\
 3.5 \\
 \hline
 6125 \\
 3675 \\
 \hline
 8)42.875 \\
 \hline
 8)5.359 \\
 .669 \\
 \hline
 \end{array}$$

$6.028 = 6 \text{ lbs. nearly.}$

2. The diameter of an iron shot is 6.7: what is its weight? *Ans. 42 lbs.*

PROBLEM II.

Having the diameter of a leaden ball given to find its weight.

RULE.

210 BALLS AND SHELLS.

R U L E.

Take $\frac{1}{3}$ of the cube of the diameter, and from it subtract $\frac{1}{3}$ of this third, and the remainder is the weight required *nearly*.

EXAMPLES.

1. What is the weight of a leaden ball, whose diameter is 3.3 inches?

$$\begin{array}{r}
 3.3 \\
 3.3 \\
 \hline
 99 \\
 99 \\
 \hline
 10.89 \\
 3.3 \\
 \hline
 3267 \\
 3267 \\
 \hline
 3)35.937 \\
 \hline
 3)11.975 \\
 3.991 \\
 \hline
 \end{array}$$

7.984 = 8 lbs. weight, *nearly*.

* The weight of a leaden ball, $4\frac{1}{2}$ inches diameter, is 17 lbs. therefore $4.25^3 : 17 \text{ lbs.} :: D^3 : \frac{17}{76.765625} \times D^3 =$
 $\frac{2}{9} D^3 \text{ nearly,} = \frac{3}{9} - \frac{1}{9} \times D^3 = \frac{1}{3} D^3 - \frac{1}{3} \text{ of}$
 $\frac{1}{3} D^3. \text{ Q. E. D.}$

Rule by Logarithms. To 3 times the log. of the diameter, add —1.3452821, and the sum will be the logarithm of the weight in pounds *exactly*.

2. What

2. What is the weight of a leaden ball whose diameter is 2.62? *Ans. 4lbs. nearly.*

PROBLEM III.

The weight of an iron ball being given, to find its diameter.

RULE.

Multiply the weight by 7, and to the product add $\frac{1}{9}$ of the weight, and the cube root of the sum will be the diameter in inches.

EXAMPLES.

1. The weight of an iron ball is 24lbs. what is the diameter?

$$\begin{array}{r} 24 \\ 7 \\ \hline 168 \\ \frac{1}{9} \text{ of } 24 = 2.666 \\ \hline 170.666 \end{array}$$

* *Demon.* By problem I. $\frac{9}{64} D^3 = \text{weight} = w$; therefore $D^3 = \frac{64}{9} \times w = \frac{63}{9} w + \frac{1}{9} w = 7w + \frac{1}{9} w$; and consequently $D = 7w + \frac{1}{9} w \sqrt[3]{1}$. Q. E. D.

The rule by Logarithms. To the log. of the weight in pounds, add 0.8519375, and $\frac{1}{3}$ of the sum will be the logarithm of the diameter in inches.

Let

† Let the supposed root be 5, whose cube is 125; then

$$\begin{array}{r}
 125 \\
 \underline{2} \\
 250 \\
 170.666 \\
 \hline
 420.666
 \end{array}
 \quad
 \begin{array}{r}
 170.666 \\
 \underline{2} \\
 341.332 \\
 125 \\
 \hline
 466.332
 \end{array}
 \quad
 \begin{array}{l}
 : \\
 5
 \end{array}
 \quad
 \begin{array}{l}
 : \\
 5
 \end{array}$$

420.666) 2331.660 (5.54 inches = cube root of
 2103330 170.666, or the dia. req.

2283300

2103330

1799700

1682664

117036

2. The weight of an iron ball is 12lbs. what is the diameter?
Ans. 4.403 inches.

PROBLEM IV.

Having the weight of a leaden ball given, to find its diameter.

† The rule here used for extracting the cube root is as follows:

Find a root nearly equal to the true one by trial, and then say, as twice the cube of the supposed root added to the given number, is to twice the given number added to the cube of the supposed root, so is the supposed root to the true root, nearly.

For a further account of this method, together with its demonstration, see the *Scholar's Guide to Arithmetic*, page 170.

R U L E.

R U L E . *

To 4 times the weight, add half the weight, and $\frac{1}{100}$ of half the weight, and the cube root of this sum will be the diameter, *nearly*.

E X A M P L E S .

1. The weight of a leaden ball is 8 lbs. what is the diameter ?

$$\begin{array}{r} 8 \\ 4 \\ \hline 32 \\ 4 = \frac{1}{2} \text{ of } 8 \\ \hline 36 \\ .12 = \frac{1}{100} \text{ of } 4 \\ \hline 36.12 \end{array}$$

Let the supposed root be 3, whose cube is 27; then

* *Demon.* By problem 2. $D^3 = \frac{76.765625}{17} \times w = 4w + \frac{1}{4}w + \frac{1}{100} \text{ of } \frac{1}{4}w$, very nearly; therefore $D = \sqrt[3]{4w + \frac{1}{4}w + \frac{1}{100} \text{ of } \frac{1}{4}w}$. Q. E. D.

The rule by Logarithms. To the log. of the weight in pounds, add 0.6547179, and $\frac{1}{3}$ of the sum will be the logarithm of the diameter, in inches, *exactly*.

$$\begin{array}{r} 27 \\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 36.12 \\ 2 \\ \hline \end{array}$$

$$\begin{array}{r} 54 \\ 36.12 \\ \hline \end{array}$$

$$\begin{array}{r} 72.24 \\ 27. \\ \hline \end{array}$$

$$\begin{array}{r} 90.12 \\ 3 \\ \hline \end{array} : 99.24 :: 3$$

90.12)297.72(3.30 = diameter, nearly.

$$\begin{array}{r} 270 \\ 36 \\ \hline \end{array}$$

$$\begin{array}{r} 27360 \\ \hline \end{array}$$

$$\begin{array}{r} 27036 \\ \hline \end{array}$$

$$\begin{array}{r} 324 \\ \hline \end{array}$$

2. What is the diameter of a leaden ball whose weight is 12 lbs?

Ans. 8.15 inches.

PROBLEM V.

Having the external and internal dimensions of an iron shell given, to find its weight.

RULE.

Take $\frac{1}{8}$ of the difference of the cubes of the diameters in inches, and $\frac{1}{8}$ of that eighth, and their sum will be the weight in pounds.

$$\begin{array}{l} \text{* Demon. The weight is } \frac{9}{64} \times D^3 - d^3 = \frac{8}{64} + \frac{1}{64} \times \\ D^3 - d^3 = \frac{1}{8} \times D^3 - d^3 + \frac{1}{8} \text{ of } \frac{1}{8} \times D^3 - d^3. \text{ Q. E. D.} \end{array}$$

EXAM-

EXAMPLES.

1. What is the weight of a 13 inch iron bomb-shell, the metal being 2 inches thick on a mean?

$$\begin{array}{r}
 13 \\
 13 \\
 \hline
 39 \\
 13 \\
 \hline
 169 \\
 13 \\
 \hline
 507 \\
 169 \\
 \hline
 \end{array}$$

2197 = cube of the external diameter.

729 = cube of the internal diameter.

$$8)1468$$

$$\begin{array}{r}
 8)183.5 \\
 22.9 \\
 \hline
 \end{array}$$

206.4 = weight required.

2. What is the weight of a 9 inch iron bomb-shell, the metal being $1\frac{1}{2}$ inches thick? *Ans.* 98.72

PROBLEM VI.

To find the number of pounds of powder that a hollow shell will hold.

RULE.

R U L E .

Cube the internal diameter in inches, and divide by 59.32, and the quotient is the weight, *nearly*.

E X A M P L E S .

1. How many pounds of powder will a hollow shell hold, whose internal diameter is 9 inches?

$$\begin{array}{r} 9 \\ 9 \\ \hline 81 \\ 9 \end{array}$$

59.32)729.00(12.28 = pounds required, *nearly*.

$$\begin{array}{r} 5932 \\ \hline \end{array}$$

$$\begin{array}{r} 13580 \\ 11864 \\ \hline \end{array}$$

$$\begin{array}{r} 17160 \\ 11864 \\ \hline \end{array}$$

$$\begin{array}{r} 52960 \\ 47456 \\ \hline \end{array}$$

$$\begin{array}{r} 5504 \end{array}$$

2. How

* *Demon.* The content of the shell is = $D^3 \times .5236$; therefore 31.06 (in. in 1lb.) : 1lb. :: $D^3 \times .5236$: $\frac{.5236}{31.06} \times D^3 =$

$$\frac{D^3}{59.32} = Q. E. D.$$

Or the rule may be expressed thus. Cube the diameter, and take $\frac{1}{60}$ part of the result; also $\frac{1}{60}$ part of this $\frac{1}{60}$; and from

2. How many pounds of powder will a hollow bomb-shell hold, whose internal diameter is 13 inches?

Aus. 37 lbs. nearly

PROBLEM VII.

To find the dimensions of a cubical box, to hold a given quantity of powder.

RULE.

Multiply the weight in pounds by 31.06, and the cube root of the product will give the length of the side in inches.

EXAMPLES.

1. What must be the length of the side of a cubical box that is to hold 15 lbs. of powder?

$$\begin{array}{r} 31.06 \\ \times 15 \\ \hline 15530 \\ 3106 \\ \hline 465.90 \end{array}$$

from the sum of these two quotients, subtract $\frac{1}{3}$ of the last quotient, and the remainder is the answer, nearly.

Rule by Logarithms. To 3 times the log. of the diameter in inches, add -2.2267989 , and the sum is the logarithm of the quantity of pounds required.

* *Demon.* As 31.06 inches : 1 lb. :: s^3 (the cube of the side) : w (the weight); whence $s = \sqrt[3]{31.06 \times w}$. Q. E. D.

Rule by Logarithms. To $\frac{1}{3}$ of the log. of the weight in pounds, add 0.4974005, and the sum will be the logarithm of the side in inches.

L

Let

Let 7.5 be the supposed root, whose cube is 421.875;
then

421.875

465.9

2

2

843.750

931.8

465.90

421.875

1309.650

:

1353.675

:: 7.5

7.5

6768375

9475725

1309.650) 10152.5625 (7.75 = side required.

9167 550

9850125

9167550

6825750

6548250

277500

2. What is the side of a cubical box, that is to
hold 12 lbs. of powder? *Ans. 7.19 inches.*

PROBLEM VIII.

Given the length, breadth, and depth of a rectan-
gular box, to find how many pounds of powder will
fill it.

RULE.

R U L E . *

Multiply the length, breadth, and depth in inches together; and this product being multiplied again by .0322 will give the answer in pounds, *nearly*.

But if greater exactness be required, subtract $\frac{1}{4000}$ part of the product of the three dimensions from the last result, and the remainder will be the answer, *very nearly*.

E X A M P L E S .

1. The length of a rectangular box is 15 inches, the breadth 13 inches, and the depth 5 inches: how many pounds of powder will it hold?

$$\begin{array}{r}
 15 \\
 13 \\
 \hline
 45 \\
 15 \\
 \hline
 195 \\
 5 \\
 \hline
 975 \\
 .0322 \\
 \hline
 1950 \\
 1950 \\
 \hline
 2925
 \end{array}$$

31.3950 = pounds required.

2. The

* *Demon.* Let L, B, D = length, breadth, and depth respectively.

Then $\frac{LBD}{31.06} = \text{weight in pounds} = LBD \times .03219574 =$

$.0322 - \frac{1}{4000} \times LBD. \quad Q. E. D.$

L 2

Rule

2. The length of a rectangular box is 1 foot, the breadth 9 inches, and the depth 4 inches: how many pounds of powder will it hold? *Ans.* 13,9104 lbs.

PROBLEM IX.

Given the diameter and length of a hollow cylinder, to find how many pounds of gunpowder will fill it.

RULE.

Multiply the square of the diameter by $\frac{1}{10}$ of the length; and then take $\frac{1}{4}$ of the product, $\frac{1}{10}$ part of this fourth, and $\frac{1}{100}$ part of this last, and the sum of all these parts will be the answer in pounds, *nearly*.

Rule by Logarithms. Add the logarithms of the length, breadth, and depth in inches, and the constant logarithm — 2.5077983 together; and the sum will be the logarithm of the answer in pounds, *exactly*.

$$\text{* Demon. } \frac{D^2 \times .78539 \times L}{31.06} = w = \frac{9}{356} \times D^2 L, \text{ nearly.}$$

$$\text{But } \frac{9}{356} = \frac{9}{360 - 4} = \frac{9}{360} + \frac{9}{360} \times \frac{4}{360} + \frac{9}{360} \times \frac{4}{360} \times \frac{4}{360}, \text{ \&c.} \\ = \frac{1}{10} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90} \text{ \&c.}$$

$$\text{Hence } w = D^2 \times \frac{L}{10} \times \left(\frac{1}{4} + \frac{1}{4} \times \frac{1}{90} + \frac{1}{4} \times \frac{1}{90} \times \frac{1}{90} \right) \text{ nearly. Q. E. D.}$$

The two first terms are near enough for common use.

Rule by Logarithms. To twice the log. of the diameter, add the logarithm of the length of the cylinder, and the constant logarithm — 2.4018885, and the sum will be the logarithm of the weight in pounds, *exactly*.

EXAMPLES.

EXAMPLES.

1. The diameter of a hollow cylinder is 4 inches, and the length 1 foot: how many pounds of powder will it hold?

$$\begin{array}{r}
 4 \\
 \hline
 4 \\
 \hline
 16 = \text{square of the diameter.} \\
 1.2 = \frac{1}{10} \text{ of the length.} \\
 \hline
 32 \\
 16 \\
 \hline
 4) 19.2 \\
 \hline
 4.8 \\
 .0533 = \frac{1}{16} \text{ of } 4.8 \\
 .0005 = \frac{1}{16} \text{ of } .053 \\
 \hline
 4.8538 \text{ pounds, the answer.}
 \end{array}$$

2. The diameter of a hollow cylinder is 6 inches, and its length 10 inches: how many pounds of powder will it hold? *Ans. 9.1 lbs.*

PROBLEM X.

Given the diameter of a hollow cylinder, to find what length of it will be filled by a given quantity of powder.

RULE.

1. Divide

$$\begin{array}{l}
 * \text{ Demon. By the last problem } L = \frac{356}{9} \times \frac{w}{D^2} = 39 \frac{5}{9} \times \\
 \frac{w}{D^2} = (40 - \frac{4}{9}) \times \frac{w}{D^2} = (40 - \frac{3}{9} - \frac{1}{9}) \times \frac{w}{D^2} = \\
 \frac{40w}{D^2} - (\frac{1}{3} \times \frac{w}{D^2} - \frac{1}{9} \text{ of } \frac{1}{3} \times \frac{w}{D^2}) \text{ nearly. Q. E. D.}
 \end{array}$$

1. Divide the weight in pounds by the square of the diameter in inches.
2. Multiply the quotient by 40, and from the product subtract $\frac{1}{3}$ of the quotient.
3. From the remainder take $\frac{1}{3}$ of that third, and this last remainder will be the answer, *nearly*.

EXAMPLES.

1. The diameter of a hollow cylinder is 1 foot: what length of it will be filled by 10 pounds of powder?

$$12 \overline{) 10}$$

$$12 \overline{) .8333}$$

$$.0694416 = 10 \text{ lbs. divided by the square of } 12$$

40

$$2.7776640$$

$$.0231472 = \frac{1}{3} \text{ of } .0694416$$

$$2.7545168$$

$$.0077157 = \frac{1}{3} \text{ of } .0231472$$

$$2.7468011 = \text{height in inches required.}$$

2. The diameter of a hollow cylinder is 6 inches: what length of it will be filled by 12 lbs. of powder?

Ans. 4.403 inches.

Rule by Logarithms. To the constant logarithm 1.5971115 add the logarithm of the given quantity of powder in pounds, and also the arithmetical complement of twice the logarithm of the diameter in inches; and the sum will be the logarithm of the length in pounds.

These rules, and their investigations, were given me by Mr. Reuben Burrow.

OF

OF THE

P I L I N G

OF

B A L L S A N D S H E L L S.

IRON shot and shells are usually piled, by horizontal courses, in a pyramidal form, the base being either an equilateral triangle, a square or a rectangle. In the triangle and square, the pile will finish in a single ball; but in the rectangle the finishing is a single row of balls.

In triangular and square piles the number of horizontal rows is always equal to the number of shot, counted on one side, in the bottom row. But, in rectangular piles, the number of courses is equal to the number of shot in the breadth of the bottom row; and the number in the top row, less one, is the difference between the number in length and breadth at the bottom.

P R O B L E M I.

To find the number of shot in a finished triangular pile.

R U L E . *

Mul-

* *Demon.* In triangular piles, each horizontal course is a triangular number, produced by taking the successive sums of the numbers 1, 2; 1, 2, 3; 1, 2, 3, 4; 1, 2, 3, 4, 5, &c. and the whole number of shot in such a pile is equal to

L 4

the

Multiply the number in the bottom row more two by that number more one; and this product being again multiplied by $\frac{1}{6}$ of the said number, will give the answer required.

EXAMPLES.

1. Required the number of shot in a finished triangular pile, the number in one side of the base being 20?

$$\begin{array}{r}
 22 = \text{number plus 2} \\
 21 = \text{number plus 1} \\
 \hline
 43 \\
 44 \\
 \hline
 462 \\
 20 \\
 \hline
 6)9240
 \end{array}$$

1540 = number of shot in the above pile.

2. Required the number of shot in a finished triangular pile, the number in one side of the base being 40? *Ans.* 11480.

the sum of all those triangular numbers, taken as far, or to as many terms, as are equal to the number in one side of the bottom course.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 3, 6, 10, 15, 21, 28, be the triangular numbers; or the number of shot in each course from the top.

But the sum of the series 1 + 3 + 6 + 10 + 15 + 21 + 28, &c. to n terms is $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ (by *Simpson's Algebra*,

page 216) $= \frac{n+2}{6} \times \frac{n+1}{2} \times n$, which is the same as the rule. Q. E. D.

PROBLEM II.

To find the number of shot in a finished square pile,

R U L E . *

Multiply the number in the bottom row more one, by the double of that number more one; and this product being again multiplied by $\frac{1}{6}$ of the said number, will give the answer required.

EXAMPLES.

1. Required the number of shot in a finished square pile, the number in one side of its base being 20.

$$40 = \text{double of } 20, \text{ plus } 1$$

$$21 = 20 \text{ plus } 1$$

$$\begin{array}{r} 41 \\ 82 \\ \hline \end{array}$$

$$82$$

$$\begin{array}{r} 41 \\ 82 \\ \hline 861 \end{array}$$

$$20$$

$$\begin{array}{r} 861 \\ 20 \\ \hline 6)17220 \end{array}$$

$$2870 = \text{number of shot in the pile.}$$

2. Required

* *Demon.* In square piles, each horizontal course is a square number, produced by taking the square of the number in its side; and the whole number of shot in such a pile is equal to the sum of all those squares, beginning at one, and proceeding as far as the number in the side of the bottom course.

Thus, if 1, 2, 3, 4, 5, 6, 7, &c. be the natural numbers, then will 1, 4, 9, 16, 25, 36, 49, &c. be the square numbers, or the number of shot in each course from the top.

2. Required the number of shot in a finished square pile, one side of the lower tier having 40 shot in it.

Ans. 22140.

PROBLEM III.

To find the number of shot in a finished rectangular pile.

RULE.

1. To twice the length of the uppermost row add the number of courses less one, and multiply the sum by the number of courses plus 1.

But the sum of the series $1+4+9+16+25+36+49$, &c. to r terms is $= \frac{n \times n+1 \times 2n+3}{6}$ (by *Simpson's Algebra*,

page 206) $= n+1 \times 2n+3 \times \frac{n}{6}$, which is the same as the rule. Q. E. D.

* The investigation of this rule is given by Mr. Simpson, in page 209 of his *Algebra* thus: Let $m+1$ and $p+1$ represent the length and breadth of the uppermost rank, and n the number of ranks one above another.

Then will $m+1 \times p+1$, $m+2 \times p+2$, $m+3 \times p+3$, &c. be the number of shot in each of the rectangular courses of the pile.

And $m+1 \times p+1 + m+2 \times p+2 + m+3 \times p+3 \dots + m+n \times p+n$ = number of shot in the pile, whether it be whole, or broken.

But $m+1 \times p+1 + m+2 \times p+2 + m+3 \times p+3 \dots + m+n \times p+n = n \times mp + \frac{n+1}{2} \times m+p + \frac{n+1 \times n+1}{6}$

$= \frac{n}{4} \times 2m+n+1 \times 2p+n+1 + \frac{n+1 \times n-1}{3}$, which is the same as the rule. Q. E. I.

2. Multiply

2. Multiply the number of courses less one by that number more one, and add $\frac{1}{2}$ of this product to the former.

3. Take $\frac{1}{2}$ of this last sum and multiply it by the number of courses, and it will give the answer required.

EXAMPLES.

1. How many shot are there in a finished rectangular pile of 15 courses, the number in the uppermost row being 32?

32

2

—

64 = twice the uppermost row.

14 = number of courses less 1.

—

78

16

—

468

78

—

1248

14 = number of courses less 1.

16 = number of courses more 1.

—

84

14

—

3)224

—

74 $\frac{2}{3}$

1248

—

1322 $\frac{2}{3}$

L 6

4)1322 $\frac{2}{3}$

$$4 \overline{) 1322\frac{2}{3}}$$

$$330\frac{2}{3}$$

$$\underline{15}$$

$$1660$$

$$\underline{330}$$

4960 = number of shot in the pile.

2. How many shot are there in a finished rectangular pile of 20 courses, the number in the top row being 40?

Ans. 11060.

The number of shot in any broken pile may be obtained, by computing what the whole finished pile would contain, and also what the pile taken away contained; for the remainder must be the number in the unfinished part as required.

The foregoing rules are expressed in algebraic terms, thus:

Let $n = N^o$ in any angular row

$n = N^o$ less one in the top row } of a Pile

$$\text{Then } \overline{n+2} \times \overline{n+1} \times \frac{n}{6} = N^o \text{ in a } \Delta$$

$$\text{And } \overline{n+1} \times \overline{2n+1} \times \frac{n}{6} = N^o \text{ in a } \square$$

$$\text{And } \overline{2n+1} + \overline{3n} \times \overline{N+1} \times \frac{n}{6} = N^o \text{ in a } \square$$

} Pile.

ARTIFICERS WORK.

ARTIFICERS estimate, or compute, the value of their works by different measures, viz.

1. *Glazing, and Mason's flat work, &c.* by the foot.

2. *Painting, Plastering, Paving, &c.* by the yard.

3. *Flooring, Partitioning, Roofing, Tiling, &c.* by the square of 100 feet.

4. *Brickwork, &c.* by the rod of $16\frac{1}{2}$ feet, whose square is $272\frac{1}{4}$.

The measures made use of in these works are contained in the following table:

12 inches	} make {	1 lineal foot
144 square inches		1 square foot
9 square feet		1 square yard
100 square feet		a square
272 $\frac{1}{4}$ square feet, or 30 $\frac{1}{4}$ square yards		1 rod, perch, or square pole

BRICKLAYERS WORK.

BRICKLAYERS compute or value their work at the rate of a brick and a half thick, and, if a wall be more or less than this standard, it must be reduced to it, as follows:

RULE.

* *Note.* In practice it is usual to divide the square feet by 27 $\frac{1}{2}$ only, omitting the $\frac{1}{2}$.

The usual way to take the dimensions of a building is to measure half round its middle, on the outside, and half round it on the inside; and this will give the true compass, in which the thickness of the wall is included.

When the height of the building is unequal, take several different altitudes, and their sum, being divided by the number you have taken, may be considered as the mean height.

To measure a chimney standing by itself, without any party-wall adjoining; girt it about for the length, and reckon the height of the story for the breadth; but if it stands against a wall, you must measure it round to the wall for the girt, and take the height as before.

When the chimney is wrought upright from the mantle-tree to the ceiling, the thickness must always be the same with the jaumbs; and nothing is ever deducted for the vacancy between the floor and the mantle-tree, because of the gathering of the breast and wings to make room for the hearth in the next story.

To measure chimney shafts, or that part which appears above the roof; girt them with a line, about the least place for the length, and take the height for the breadth; and if they be four inches thick, set down the thickness at one brick-work; but if they be nine inches thick, reckon it at a brick and a half, in consideration of the plastering and scaffolding.

All

RULE.

Multiply the superficial content of the wall in feet, by the number of half bricks in the thickness, and $\frac{1}{3}$ of that product will be the content required.

EXAMPLES.

1. How many square rods are there in a wall $52\frac{1}{2}$ feet long, 12 feet 9 inches high, and $2\frac{1}{2}$ bricks thick?

By Decimals.

$$\begin{array}{r} 52.5 \text{ length} \\ 12.75 = \text{height} \end{array}$$

2625

3675

1050

525

272)669.375(2.4609

544

5 half bricks

1253 3)12.3045

1088

ro. fr. in.

4.1015 = 4 27 7ths answer.

1657

1632

2550

2448

102

All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the value

of

By Cross Multiplication.

feet in.

52 6

12 9

630 0

39 4 6

272)669 4 6

2 124 4

5

3)12 82 8

4 27

7 as before.

2. How many square rods are there in a wall $62\frac{1}{2}$ feet long, 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick?

ro. sq. in. p.

Ans. 5 167 9 4

3. If each side wall of a building be 45 feet long on the outside, each end wall 15 feet broad on the inside, the height of the building 20 feet, and the gable at each end of the wall 6 feet high, the whole being 2 bricks thick; what is the true content in standard rods?

Ans. 12.1761.

of their workmanship is added to the bill at the stated rate agreed on.

There are also other allowances to be made to the workman, such as those for returns or angles made by two adjoining walls, and double measure for feathered gable ends, &c.

M A S O N S W O R K.

TO Masonry belongs all sorts of stone work, and the measure made use of is a foot, either superficial or solid.

Walls, blocks of marble or stone, columns, &c. are measured by the solid foot; and pavements, slabs, chimney-pieces, &c. by the superficial foot.*

EXAMPLES.

1. Required the solid content of a wall whose length is 48 feet 6 inches, its height 10 feet 9 inches, and thickness 2 feet.

By Decimals.

$$\begin{array}{r}
 48.5 \\
 10.75 \\
 \hline
 2425 \\
 3395 \\
 4850 \\
 \hline
 521.375 \\
 2 \\
 \hline
 \hline
 \end{array}$$

1042.750 the answer.

* Solid measure is principally used for materials, and the superficial for workmanship.---In the solid measure the true length, breadth and thickness, are taken, and multiplied continually together. And in the superficial measure, the length and breadth of every part of the projection must be taken, which appears without the general upright face of the building.

By

By Cross Multiplication,

feet in.

48 6

10 9

485 0

36 4 6

521 4 6

2 0 0

1042 9 0 *the same as before.*

2. Required the solid content of a wall whose length is 53 feet 6 inches, its height 12 feet 3 inches, and its thickness 2 feet? *Ans. 1310 feet 9 in.*

3. What is a marble slab worth, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches, at 6s. per foot? *Ans. 3l. 15. 5d.*

4. What is the solid content of a wall, whose length is 60 feet 9 inches, its height 10 feet 3 inches, and its thickness $2\frac{1}{2}$ feet? *Ans. 1556.71875 feet.*

5. In a chimney-piece, suppose the
Length of the mantle and slab each, - - - - - 4 6 }
Breadth of both together, - - - - - 3 2 }

Length of each jamb, - - - - - 4 4 }
Breadth of both together, - - - - - 1 9 }

What will be the content of the chimney-piece?

Ans. 21 feet 10 in.

OF

CARPENTERS AND JOINERS WORK.

CARPENTERS and Joiners work is that of flooring, partitioning, roofing, &c. and is measured by the square of 100 feet.

E X A M -

* *Note.* Large and plain articles are usually measured by the foot, or yard, &c. square; but enriched mouldings, and some other articles, are often estimated by running or lineal measure, and some things are rated by the piece.

In measuring of joists it is to be observed, that only one of their dimensions is the same with that of the floor, and the other will exceed the length of the room by the thickness of the wall, and $\frac{1}{2}$ of the same, because each end is let into the wall about $\frac{3}{4}$ of its thickness.

No deductions are made for hearths, on account of the additional trouble and waste of materials.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In measuring of Joiners work, the string is made to ply close to every part of the work over which it passes.

The measure of centering for cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length; but in groin-centering, it is usual to allow double measure, on account of their extraordinary trouble.

In roofing, the length of the house in the inside, together with $\frac{3}{4}$ of the thickness of one gable, is to be considered as the length, and the breadth is equal to double the length of a string which is stretched from the ridge down to the rafter, along the eaves-board, till it meets with the top of the wall.

For

236 CARPENTERS AND JOINERS WORK.

EXAMPLES.

1. If a floor be 57 feet, 3 inches long, and 28 feet 6 inches broad: how many squares will it contain?

By Decimals.

$$\begin{array}{r}
 57.25 \\
 28.5 \\
 \hline
 28625 \\
 45800 \\
 11450 \\
 \hline
 1631.625
 \end{array}$$

By

For fair-cases. Take the breadth of all the steps, and make a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends, and by the breadth is to be understood the girt of its two upper surfaces, or the tread and riser.

For the balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel post, for the length; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the breadth.

For wainscoting, take the compass of the room for the length; and the height from the floor to the cieling, making the string ply close into all the mouldings, for the breadth.--- Out of this must be made deductions for windows, doors, and chimneys, &c. but workmanship is counted for the whole, on account of the extraordinary trouble.

For doors, it is usual to allow for their thickness, by adding it into both the dimensions of length and breadth, and then multiplying them together for the area.---If the door be pannelled on both sides, take double its measure for the workmanship; but if one side only be pannelled, take the area and its

By Cross Multiplication.

$$\begin{array}{r}
 \text{feet in.} \\
 57 \quad 3 \\
 28 \quad 6 \\
 \hline
 1603 \quad 0 \\
 28 \quad 7 \quad 6 \\
 \hline
 16131 \quad 7 \quad 6
 \end{array}$$

Ans. 16 squares, and 31 feet.

2. A floor is 53 feet 6 inches long, and 47 feet 9 inches broad: how many squares will it contain?

Ans. 25 sq. and 54 feet.

3. A partition is 91 feet 9 inches long, and 11 feet 3 inches broad: how many squares will it contain?

Ans. 10 sq. and 32 feet.

4. If a house within the walls be 44 feet 6 inches long, and 18 feet 3 inches broad: how many squares of roofing will cover it?

Ans. 12 sq. and 18 feet.

5. If a house measure within the walls 52 feet 8 inches in length, and 30 feet 6 inches in breadth, and the roof be of a true pitch: what will it cost roofing at 10s. 6d. per square?

Ans. 12l. 12s. 11½d.

its half for the workmanship.---For the surrounding architrave, gird it about the outermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

Window-butters, bases, &c. are measured in the same manner.

In the measuring of roofing for workmanship alone, all holes for chimney shafts and sky-lights are generally deducted.

But in measuring for work and materials, they commonly measure in all sky-lights, luthern-lights, and holes for the chimney shafts, on account of their trouble and waste of materials.

SLATERS AND TILERS WORK.

IN these works, the content of a roof is found by multiplying the length of the ridge by the girt from eve to eve; and, in slating, allowance must be made for the double row at the bottom.

Double measure is generally allowed for hips, vallies, gutters, &c. but no deductions are made for chimnies.

EXAMPLES.

1. The length of a slated roof is 45 feet 9 inches, and its girt 34 feet 3 inches: what is the content?

By Decimals.

$$\begin{array}{r}
 45.75 \\
 34.25 \\
 \hline
 22875 \\
 9150 \\
 18300 \\
 13725 \\
 \hline
 9)1566.9375 \\
 \hline
 174.104
 \end{array}$$

By

By Cross Multiplication.

<i>feet</i>	<i>in.</i>
45	9
34	3
1555	6
11	5 3
9)1566	11 3
174 0	11 3

Ans. 174 yards.

2. What will the tiling a barn cost at 25s. 6d. per square, the length being 43 feet 10 inches, and the breadth 27 feet 5 inches, on the flat, the eave-boards projecting 16 inches on each side? *Ans.* 24l. 9s. 5½d.

In angles formed in a roof, running from the ridge to the eaves, that angle of the roof, which bends inwards, is called a *valley*; and the angle bending outwards is called a *bip*. And in tiling and slating, it is common to add the length of the vallies to the content in feet; and sometimes also the hips are added.

In slating it is common to reckon the breadth of the roof 2 or 3 inches broader than what it measures, because the first row is almost covered by the second; and this is done sometimes when a roof is tiled.

Note. Skylights and chimney-shafts are always deducted; but they seldom deduct luthern lights, or garret windows on the roof; for the covering them is reckoned equal to the hole in the roof.

PLASTERERS

PLASTERERS WORK.

PLASTERERS work is of two kinds, viz. plaistering upon laths, called cieling; and plaistering upon walls, called rendering; and these different kinds must be measured separately, and their contents collected into one sum.

Note. Proper deductions must be made for doors, windows, &c.—And in measuring between quarters, there is commonly $\frac{1}{3}$ part of the whole area allowed; but when rendering between quarters is whitened or coloured, there is $\frac{1}{3}$ part to be added to the whole for the sides of the quarters and braces.

EXAMPLES.

If a cieling be 59 feet 9 inches long, and 24 feet 6 inches broad; how many yards does it contain?

By Decimals.

59.75

24.5

29875

23900

11950

9)1463.875

162.652

By

By Cross Multiplication.

<i>feet</i>	<i>in.</i>
59	9
24	6
1434-	0
29	10 6
9)1453	10 6
162	5 10 6

Ans. 162 yards 5 feet.

2. If the partitions between rooms be 141 feet 6 inches about, and 11 feet 3 inches thick; how many yards do they contain? *Ans.* 176.87.

3. There is a quantity of partitioning that measures 234 feet 8 inches about, and 14 feet 6 inches high, and is rendered between quarters; the lathing and plastering of which will be 8*d.* per yard, and the whitening 2*d.* per yard: what will the whole come to? *Ans.* 13*l.* 14*s.* 6*d.*

4. The length of a room is 14 feet 5 inches, its breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, whose girth is 8½ inches, and its projection 5 inches from the wall on the upper part next the ceiling: what will be the quantity of plastering, supposing there are no deductions but for one door, whose size is 7 feet by 4 feet? *Ans.* 53 yards 5 feet 3 inches, of rendering, 18 yards 5 feet 6 inches, of ceiling, and 39 feet 0 ⅙ inches, of cornice.

P A I N T E R S W O R K.

PAINTERS work is measured in the same manner as that of Joiners; and in taking the dimensions, the line must be forced into all the mouldings and corners.

Windows are done at so much a-piece, and in carved mouldings, &c. it is customary to allow double the usual measure.

E X A M P L E S.

1. If a room be painted, whose height is 16 feet 6 inches, and its compass 97 feet 9 inches, how many yards does it contain?

By Decimals.

$$\begin{array}{r} 97.75 \\ 16.5 \\ \hline \end{array}$$

$$\begin{array}{r} 48875 \\ 58650 \\ 9775 \\ \hline \end{array}$$

$$\begin{array}{r} 9)1612.875 \\ \hline \end{array}$$

$$179.208$$

By

By Cross Multiplication.

feet	in.	
97	9	
16	6	
1564	0	
48	10	6
9)1612	10	6
179	1	10 6

Ans. 179 yards 1 foot.

2. The height of a room is 14 feet 10 inches, and the circumference 21 feet 8 inches: how many square yards does it contain? *Ans.* 35.

3. Suppose a room, that was to be painted at 8*d.* per yard, measures as follows: The height is 11 *fe.* 7 *in.* the girt or compass 74 *fe.* 10 *in.* the door 7 *fe.* 6 *in.* by 3 *fe.* 9 inches; five window shutters, each 6 *fe.* 8 *in.* by 3 *fe.* 4 *in.* the breaks in the windows 14 *in.* deep, and 8 *fe.* high; the chimney 6 *fe.* 9 *in.* by 5 *fe.* a closet, the height of the room, 3½ *fe.* deep, and 4¾ *fe.* in front, with shelving, at 22 *fe.* 6 *in.* by 10 *in.* the shutters, door and shelves, being all coloured on both sides: what will the whole come to?

Ans. 4*l.* 18*s.* 9*d.*

Note. Painters take their dimensions with a string, and measure from the top of the cornice to the floor, girting the string over all the mouldings and swellings; and their price is generally proportioned to the number of times they lay on their colour.

GLAZIERS WORK.

GLAZIERS take their dimensions in feet, inches, and parts, and estimate their work by the square foot.

In taking the length and breadth of a window, the cross-bars between the panes are always included.

Windows of every form are measured as though they were squares, and the greatest lengths and breadths are constantly to be taken, on account of the waste attending the cutting.

EXAMPLES.

1. If a pane of glass be 2 feet 8 inches and 3 quarters long, and 1 foot 4 inches 1 quarter broad: how many feet does it contain?

By Decimals.

The decimal of $8\frac{3}{4}$ inches is .729

And that of $4\frac{1}{4}$ inches is .354

2.729

1.354

10916

13645

8187

2729

3.695066

12

8.340792

By

By Cross Multiplication.

feet in.

2 8 9

1 4 3

—

2 8 9

10 11 0

8 2 3

—

3 8 5 2 3

Ans. 3 feet 8 in. $\frac{5}{10}$.

2. What will the glazing a triangular sky-light come to at 10*d.* per foot, supposing the base to be 12 feet 6 inches, and the perpendicular height 16 feet 9 inches?

Ans. 4*l.* 7*s.* 2 $\frac{3}{4}$ *d.*

3. There is a house with 3 tier of windows, three in a tier; the height of the first tier is 7 feet 10 inches; of the second 6 feet 8 inches; of the third 5 feet 4 inches, and the breadth of each 3 feet 11 inches: what will the glazing come to at 14*d.* per foot?

Ans. 13*l.* 11*s.* 10 $\frac{1}{2}$ *d.*

4. There is a house with 3 tier of windows, three in a tier; the height of those in the lower tier being 7 *fe.* 9 *in.* of those in the middle 6 *fe.* 6 *in.* and of those in the upper tier 5 *fe.* 3 $\frac{1}{2}$ *in.* and having also a circular window above the door, the greatest height of which is 1 foot 10 $\frac{1}{2}$ *in.* and the common breadth of all the windows 3 *fe.* 9 *in.* what will the glazing come to at 13 $\frac{1}{2}$ *d.* per foot?

Ans. 12*l.* 5*s.* 6*d.*

Note. Windows are sometimes measured by taking the dimensions of one pane, and multiplying it continually by the number of panes; and no allowance is ever made for round or oval windows, as the trouble of cutting them to those shapes is more than the value of the glass omitted.

O F

P A V I O U R S W O R K . *

PAVIOURS work is done by the square yard, and the content is found by multiplying the length by the breadth.

E X A M P L E S .

1. What will the paving a rectangular court-yard come to at 31. 2d. per yard, supposing the length to be 27 feet 10 inches, and the breadth 14 feet 9 inches?

By Cross Multiplication.

	feet	in.	
	27	10	
	14	9	
	389	8	
	20	10	6
9)	410	6	6
	45	5	6 6

* Plumbers work is generally done by the pound, or hundred weight.

s.	d.		45	5	6	6 at 3s. 2d.
2	0	$\frac{1}{10}$	45			
1	0	$\frac{1}{2}$	4	10		
	2	$\frac{1}{6}$	2	5		
f.	in.	p.				
3	0	0		7	6	
1	0	0		1	$0\frac{1}{2}$	
1	0	0		0	4	
0	6	0		0	4	
0	0	6		0	2	
		$\frac{1}{12}$		0	0	
			7	4	$4\frac{1}{2}$	the answer.

2. A rectangular court-yard is 42 feet 9 inches long, and 68 feet 6 inches in depth, and a foot-way goes quite through it, of 5 feet 6 inches in breadth; the foot-way is laid with stone at 3s. 6d. per yard, and the rest with pebbles at 6s. per yard: what will the whole come to? *Ans.* 49l. 17s. $0\frac{1}{4}$ d.

Sheet-lead, used in roofing, guttering, &c. is generally between 7 and 12lbs. weight to the square foot.

The following table will shew the weight of a square foot to each of these thicknesses.

Thick.	lbs. sq. foot.	Thick.	lbs. sq. foot.	Thick.	lbs. sq. foot.
$\frac{1}{8}$	7.373	.15	8.848	.18	10.618
.13	7.658	.16	9.438	.19	11.207
.14	8.258	$\frac{1}{8}$	9.831	$\frac{1}{2}$	11.797
$\frac{1}{7}$	8.427	.17	10.028	.21	12.387

VAULTED AND ARCHED ROOFS.

ARCHED roofs are either *vaults*, *domes*, *saloons*, or *groins*.

Vaulted roofs are formed by arches springing from the opposite walls, and meeting in a line at the top.

Domes are made by arches springing from a circular or polygonal base, and meeting in a point at the top.

Saloons are formed by arches connecting the side walls to a flat roof, or ceiling, in the middle.

Groins are formed by the intersection of vaults with each other.

Vaulted roofs are generally one of the three following sorts:

1. *Circular roofs*, or those whose arch is some part of the circumference of a circle.
2. *Elliptical roofs*, or those whose arch is some part of the circumference of an ellipsis.
3. *Gothic roofs*, or those which are formed by two circular arcs that meet in a point directly over the middle of the breadth, or span of the arch.

PROBLEM I.

To find the solid content of circular, elliptic, or gothic vaulted roofs.

RULE.

R U L E . *

Multiply the area of one end by the length of the roof, and the product will be the solidity required.

E X A M P L E S .

1. What is the solid content of a semi-circular vault, whose span is 40 feet, and its length 120 feet?

$$\begin{array}{r} .7854 \\ 1600 = \text{square of } 40. \end{array}$$

$$\begin{array}{r} 4712400 \\ 7854 \end{array}$$

$$2)1256.64 = 0$$

$$628.32 = \text{area of the end.}$$

$$120 = \text{length.}$$

$$75398.40 = \text{solidity required.}$$

2. Required the solidity of an elliptic vault, whose span is 40 feet, height 12 feet, and length 80?

$$\text{Ans. } 30159.36 \text{ feet.}$$

3. What is the solid content of a gothic vault, whose span is 48, the chord of its arch 48, the distance of the arch from the middle of the chord 18, and the length of the vault 18? *Ans. 136228.044.*

* To find the solidity of the materials in either of the arches.

Rule. From the solid content of the whole arch take the solid content of the void space, and the remainder will be the solidity of the arch.

250 VAULTED AND ARCHED ROOFS.

PROBLEM II.

To find the concave, or convex surface, of circular, elliptic, or gothic vaulted roofs.

R U L E.*

Multiply the length of the arch by the length of the vault, and the product will be the superficies required.

EXAMPLES.

1. What is the concave surface of a semi-circular vault, whose span is 40 feet, and its length 120?

$$\begin{array}{r} 3.1416 \\ 40 \\ \hline \end{array}$$

$$2)125.6640$$

$$\begin{array}{r} 62.832 \\ 120 \\ \hline \end{array} = \text{length of the arch.}$$

$$7539.840 = \text{concave surface required.}$$

PROBLEM III.

To find the solid content of a dome; its height, and the dimensions of its base being known.

* The convex surface of a vault may be found by stretching a string over it; but for the concave surface this method is not applicable, and therefore its length must be found from proper dimensions.

R U L E.

R U L E. *

Multiply the area of the base by $\frac{2}{3}$ of the height, and the product will be the solidity.

E X A M P L E S.

1. What is the solid content of a spherical dome, the diameter of whose circular base is 60 feet?

$$\begin{array}{r} .7854 \\ 3600 = \text{square of } 60. \\ \hline \end{array}$$

$$\begin{array}{r} 4712400 \\ 23562 \\ \hline \end{array}$$

$$\begin{array}{r} 2827.4400 = \text{area of the base.} \\ 20 = \frac{2}{3} \text{ of the height (30)} \\ \hline \end{array}$$

$$56548.8000 = \text{solidity required.}$$

2. In an hexagonal spherical dome, one side of the base is 20 feet; what is the solidity?

Ans. 12000 feet.

P R O B L E M I V.

To find the superficial content of a spherical dome.

R U L E. †

* Domes and saloons are of various figures, but they are things that seldom occur in the practice of measuring.

† The practical rule for elliptical domes is as follows:

Rule. Add the height to half the diameter of the base, and this sum multiplied by 1.5708 will give the superficial content nearly.

M 6-

Multiply

252 VAULTED AND ARCHED ROOFS.

Multiply the area of the base by 2, and the product will be the superficial content required.

EXAMPLES.

1. What will the painting an hexagonal spherical dome come to at 1s. per yard; each side of the base being 20 feet?

$$2.598076 = \text{area of a hexagon whose side is 1.}$$

$$400 = \text{square of 20.}$$

$$1039.230400 = \text{area of the base.}$$

$$\quad \quad \quad 2$$

$$9)2078.460800 = \text{superficial content required.}$$

$$2.0)230.940088$$

$$11.5470044 = 11l. 10s. 11d. \text{ the expence of } \left[\begin{array}{l} \text{painting.} \\ \text{of} \end{array} \right.$$

PROBLEM V.

To find the solid content of a saloon.

R U L E. *

1. Multiply

* To find the superficial content of a saloon.

1. Find the area of the flat part of the cieling.
2. Find the convex surface of a cylinder, or cylindroid, whose length is equal to $\frac{1}{2}$ the perimeter of the cieling, and its diameters to twice the height and twice the projection of the arch.
3. Find the superficial content of a dome of the same figure as the arch, and whose base is either a square, or a figure similar to that of the cieling; the side being equal to the difference of a side of the room, and a side of the cieling.

4. Add

1. Multiply the height of the arc, its projection, $\frac{1}{2}$ of the perimeter of the cieling, and 3.1416 continually together, and call the product A.

2. From a side or diameter of the room take a like side or diameter of the cieling, and multiply the square of the remainder by the proper factor (page 54) and this product again by $\frac{2}{3}$ of the height, and call the last product B.

3. Multiply the area of the flat cieling by the height of the arch, and this product added to the sum of A and B will give the content required.

EXAMPLES.

1. What is the solid content of a saloon with a circular quadrantal arch of 2 feet radius, springing over a rectangular room of 20 feet long and 16 feet wide?

Here the flat part of the cieling is 16 feet by 12; and

$$\begin{array}{r} 4)56 \\ \hline \end{array}$$

14 = $\frac{1}{2}$ of the perimeter.

2 = height.

$$\begin{array}{r} \hline 28 \end{array}$$

2 = projection.

$$\begin{array}{r} \hline 56 \end{array}$$

4. Add these three articles together, and the sum will give the superficial content required.

Note. In a rectangular, circular or polygonal room, the base of the dome will be a square, a circle, or a like polygon.

$$3.1416$$

254 VAULTED AND ARCHED ROOFS

$$\begin{array}{r}
 311416 \\
 56 \\
 \hline
 188496 \\
 157080 \\
 \hline
 175.9296 = A.
 \end{array}$$

20 = side of the room.

16 = side of the ceiling.

4

4

16

1.000 &c. = factor.

16.000

$1\frac{1}{2} = \frac{3}{2}$ of the height.

16.000

5.333

21.333 = B.

16

12

192 = area of the flat ceiling.

2 = height of the arch.

384

175.9296

21.3333

581.2629 = solid content required.

2. A cir.

2. A circular building of 40 feet diameter, and 25 feet high to the cieling, is covered with a saloon, whose circular arch is 5 feet radius; required the capacity of that room in cubic feet?

Ans. 30779.45948 feet.

PROBLEM VI.

To find the solid content of the vacuity formed by a groin arch, either circular or elliptical.

R U L E.

Multiply the area of the base by the height, and the product again by .904, and it will give the solidity required.

EXAMPLES.

1. What is the solid content of the vacuity formed by a circular groin, one side of its square base being 12 feet?

12

12

144 = area of the base.

6 = height.

864

.904

3456

77760

781.056 = solidity required.

2. What

256 VAULTED AND ARCHED ROOFS.

2. What is the solid content of the vacuity formed by an elliptical groin, one side of its square base being 20 feet, and the height 6 feet? *Ans.* 2169.6.

PROBLEM VII.

To find the concave superficies of a circular groin.

RULE.*

Multiply the area of the base by 1.1416, and the product will be the superficies required.

EXAMPLES.

1. What is the curve superficies of a circular groin arch, one side of its square base being 12 feet?

$$\begin{array}{r}
 12 \\
 12 \\
 \hline
 144 = \text{area of the base.} \\
 1.1416 \\
 \hline
 45664 \\
 45664 \\
 11416 \\
 \hline
 164.3904 = \text{superficies required.}
 \end{array}$$

2. What

* This rule may also be observed in elliptical groins, the error being too small to be regarded in practice.

In measuring works where there are many groins in a range, the cylindrical pieces between the groins, and on their sides, must be computed separately.

And

2. What is the concave superficies of a circular groin arch, one side of its square being 9 feet?

Ans. 92.4696.

And to find the solidity of the brick, or stone-work, which forms the groin arches, observe the following

Rule. Multiply the area of the base by the height, including the work over the top of the groin, and this product lessened by the solid content, found as before, will give the solidity required.

The general rule for measuring all arches is this:

From the content of the whole, considered as solid, from the springing of the arch to the outside of it, deduct the vacuity contained between the said springing and the under side of it, and the remainder will be the content of the solid part.

And because the upper sides of all arches, whether vaults or groins, are built up solid, above the haunches, to the same height with the crown, it is evident that the area of the base will be the whole content above-mentioned, taking for its thickness the height from the springing to the top. And for the content of the vacuity to be deducted, take the area of its base, accounting its thickness to be $\frac{2}{3}$ of the greatest inside height. But it may be noted that the area used in the vacuity, is not exactly the same with that used in the solid; for the diameter of the former is twice the thickness of the arch less than that of the latter.

And although I have mentioned the deduction of the vacuity as common to both the vault and the groin, it is reasonable to make it only in the former, on account of the waste of materials and trouble to the workman, in cutting and fitting them for the angles and interseptions.

Whoever wishes to see this subject more fully handled, may consult *La Théorie et la Pratique de la Géométrie*, par M. l'Abbé Deidier.

OF THE
CARPENTER'S RULE.

THIS instrument is commonly called *Cogeshall's* sliding rule. It consists of two pieces, of a foot in length each, which are connected together by means of a folding joint.

On one side of the rule, the whole length is divided into inches and half quarters, for the purpose of taking dimensions. And on this face there are also several plain scales, divided by diagonal lines into twelfth parts, which were designed for planning such dimensions as are taken in feet and inches.

On one part of the other face there is a slider, and four lines marked A, B, C, and D; the two middle ones B and C being upon the slider.

Three of these lines A, B, C, are double ones, because they proceed from one to 10 twice over; and the fourth line D is a single one, proceeding from 4 to 40; and is called the *girt line*.

The use of the double lines, A and B, is for working proportions, and finding the areas of plane figures. And the use of the girt line D, and the other double line C, is for measuring solids.

When 1 at the beginning of any line is counted 1, then the 1 in the middle will be 10, and the 10 at the end 100. And when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000, &c. and all the small divisions are altered in value accordingly.

Upon

Upon the other part of this face there is a table of the value of a load of timber, at all prices, from 6*d.* to 2*s.* a foot.

Some rules have likewise a line of inches, or a foot divided decimally into 10th. parts; as well as tables of board measure, timber measure, &c. but these will be best understood from a sight of the instrument.

The use of the SLIDING RULE.

PROBLEM I.

To find the product of two numbers, as 7 and 26.

RULE.

Set 1 upon A, to one of the numbers (26) upon B; then against the other number (7) on A, will be found the product (182) upon B.

Note. If the third term runs beyond the end of the line, seek it on the other radius, or part of the line, and increase the product 10 times.

PROBLEM II.

To divide one number by another, as 510 by 12.

RULE.

Set the divisor (12) on A, to 1 on B; then against the dividend (510) on A, is the quotient ($42\frac{1}{2}$) on B.

Note. If the dividend runs beyond the end of the line, diminish it 10 or 100 times to make it fall on A, and increase the quotient accordingly.

PRO-

260 CARPENTER'S RULE.

PROBLEM III.

To square any number, as 27.

RULE.

Set 1 upon D to 1 upon C; then against the number (27) upon D, will be found the square (729) upon C.

If you would square 270, reckon the 1 on D to be 100; and then the 1 on C will be 1000, and the product 72900.

PROBLEM IV.

To extract the square root of any number, as 4268.

RULE.

Set 1 upon C, to 1 upon D; then against (4268) the number on C, is (65.3) the root on D.

To value this right you must suppose the 1 on C to be some of these squares 1, 100, 10000, &c. which is the nearest to the given number, and then the root corresponding will be the value of the 1 upon D.

PROBLEM V.

To find a mean proportional between any two numbers, as 27 and 450.

RULE.

Set one of the numbers (27) on C, to the same on D; then against the other number (450) on C, will be the mean (112) on D.

Note. If one of the numbers overruns the line, take the 100th. part of it, and augment the answer 10 times.

PRO-

PROBLEM VI.

Three numbers being given to find a fourth proportional; suppose 12, 28 and 57.

RULE. *

Set the first number (12) upon A, to the second (28) upon B; then against the third number (57) on A, is the fourth (133) on B.

Note. If one of the middle numbers runs off the line, take the 10th. part of it only, and augment the answer 10 times.

The finding a third proportional is the same, the second number being twice repeated.

* The use of the rule in board and timber measure will be shewn in what follows.

If the breadth of a board be given; to find how much in length will make a square foot.

Rule. If the board be narrow, it will be found in the table of board measure on the rule; but, if not, shut the rule, and seek the breadth in the line of board measure, running along the rule, from that table; then over against it, on the opposite side, is the length in inches required.

The side of the square of a piece of timber being given; to find how much in length will make a foot solid.

RULE. If the timber be small, it will be found in the table of timber measure on the rule; but, if not, look for the side of the square, in the line of timber measure, running along the rule, from that table, and against it in the line of inches is the length required.

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OF

TIMBER MEASURE.

PROBLEM I.

To find the area, or superficial content, of a board or plank.

RULE.

MULTIPLY the length by the breadth, and the product will be the content required.

Note. When the board is tapering, add the breadths of the two ends together, and take $\frac{1}{2}$ the sum for the mean breadth.

By the SLIDING RULE.

Set 12 on B to the breadth in inches on A, then against the length in feet on B, is the content on A, in feet and fractional parts, as required.

EXAMPLES.

1. What is the value of a plank, whose length is 8 feet 6 inches, and breadth 1 foot 3 inches throughout, at $2\frac{1}{2}$ d. per foot?

<i>feet in.</i>	
8	6
1	3
8	6
2	1 6
10	7 6 <i>the content.</i>

$$\begin{array}{r}
 10 \quad 7 \quad 6 \\
 \hline
 2d. \text{ is } \frac{1}{8} \quad 1 \quad 8 \\
 \frac{1}{2} \text{ is } \frac{1}{4} \quad 5 \\
 in. \\
 6 \text{ is } \frac{1}{2} \quad 1 \frac{1}{4} \\
 in. \\
 1 \text{ is } \frac{1}{8} \quad \frac{1}{4} \\
 \hline
 2s. \quad 2 \frac{1}{2}d. \text{ the answer.}
 \end{array}$$

By the SLIDING RULE.

in.
As 12 on B : 15 on A :: $8\frac{1}{2}$ on B : $10\frac{1}{2}$ on A.

2. What is the content of a board, whose length is 5 feet 7 inches, and breadth 1 foot 10 inches?

feet in.
Ans. 10 2 10

3. At $1\frac{1}{2}$ per foot, what is the value of a plank, whose length is 12 feet 6 inches, and breadth 11 inches throughout?

Ans. 1s. 5d.

4. Find the value of 5 oaken planks at 3d. per foot, each being $17\frac{1}{2}$ feet long, and their particular breadths as follows: viz. two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{2}$ at the narrower? *Ans.* 1l. 5s. $8\frac{1}{4}d.$

PROBLEM II.

To find the solidity of squared or four-sided timber.

RULE.

R U L E. *

Multiply the mean breadth by the mean thickness, and this product again by the length, and it will give the solidity required.

By the SLIDING RULE.

As the length in feet on *C* : 12 on *D* :: quarter girt in inches on *D* : solidity on *C*.

* *Note 1.* If the stick be equally broad and thick throughout, the breadth and thickness, any where taken, will be the mean breadth and thickness.

2. If the tree tapers regularly from one end to the other, the breadth and thickness, taken in the middle, will be the mean breadth and thickness.

3. If the stick does not taper regularly, but is thicker in some places than in others, let several different dimensions be taken, and their sum divided by the number of them will give the mean dimensions.

This method of finding the mean dimensions is mostly used in practice, but, in many cases, it is exceedingly erroneous.

The quarter girt, likewise, which is mentioned in the proportion by the sliding rule, is subject to error. It is not the fourth part of the circumference, but the square root of the product arising from multiplying the mean breadth by the mean thickness.

In order to shew the fallacy of taking $\frac{1}{4}$ of the girt for the side of a mean square, take the following example:

Suppose a piece of timber to be 24 feet long, and a foot square throughout, and let it be slit into two equal parts, from end to end.

Then the sum of the solidities of the two parts, by the quarter girt method, will be 27 feet, but the true solidity is 24 feet; and if the two pieces were very unequal, the difference would be still greater.

E X A M P L E S.

EXAMPLES.

1. The length of a piece of timber is $20\frac{1}{2}$ feet, the breadth at the greater end is 1 foot 9 inches, and the thickness 1 foot 3 inches; and at the lesser end the breadth is 1 foot 6 inches, and the thickness 1 foot: what is the solidity?

$$1.75 = \text{greater breadth.}$$

$$1.5 = \text{lesser breadth.}$$

$$\begin{array}{r} 2)3.25 \\ \hline \end{array}$$

$$1.625 = \text{mean breadth.}$$

$$1.25 = \text{greater thickness.}$$

$$1.00 = \text{lesser thickness.}$$

$$\begin{array}{r} 2)2.25 \\ \hline \end{array}$$

$$1.125 = \text{mean thickness.}$$

By Decimals.

$$1.625$$

$$1.125$$

$$\hline$$

$$8125$$

$$3250$$

$$1625$$

$$1625$$

$$\hline$$

$$1.828125$$

$$20.5$$

$$\hline$$

$$9140625$$

$$36562500$$

$$\hline$$

$$37.4765625 = \text{content.}$$

N

By

TIMBER MEASURE.

By Cross Multiplication.

$$\begin{array}{r}
 \text{fo. in.} \\
 1 \ 7 \ 6 \\
 1 \ 1 \ 6 \\
 \hline
 1 \ 7 \ 6 \\
 7 \ 6 \\
 9 \ 9 \\
 \hline
 1 \ 9 \ 11 \ 3 \\
 20 \ 6 \\
 \hline
 36 \ 6 \ 9 \ 0 \\
 10 \ 11 \ 7 \ 6 \\
 \hline
 37 \ 5 \ 8 \ 7 \ 6 = \text{content.}
 \end{array}$$

By the SLIDING RULE.

As 1 upon B : $19\frac{6}{12}$ upon A :: $13\frac{6}{12}$ upon B : $263\frac{3}{8}$ upon A, the mean square.

As 16 upon C : 4 upon D :: 1.8 upon C : 16.2 upon D, the side of the mean square.

As $20\frac{1}{2}$ upon C : 12 upon D :: 16.2 upon D : $37\frac{1}{2}$ upon C, the answer.

2. The length of a piece of timber is 24.5 feet, and its ends are equal squares, whose sides are each 1.04 feet: what is the solidity?

Ans. 26 feet 6 inches.

3. The length of a piece of timber is 20.38 feet, and the ends are unequal squares, the side of the greater being $19\frac{1}{2}$ inches, and that of the lesser $9\frac{1}{2}$ inches: what is the solidity?

Ans. 29 feet, $8\frac{1}{2}$ inches.

4. The length of a piece of timber is 27.36 feet; at the greater end the breadth is 1.78 feet, and the thickness 1.23 feet; and at the lesser end the breadth is 1.04 feet, and the thickness .91 feet: what is the solidity?

Ans. 41 feet, $2\frac{1}{2}$ inches.

P R O.

PROBLEM III.

To find the solidity of round or unsquared timber.

RULE I.*

Multiply the square of the quarter girt (or $\frac{1}{4}$ of the circumference) by the length, and the product will be the content, according to the common practice.

By the SLIDING RULE.

As the length upon C : 12 upon D :: $\frac{1}{4}$ girt upon D : content upon C.

EXAMPLES.

1. A piece of timber is $9\frac{1}{2}$ feet long, and the quarter girt is 39 inches : what is the solidity ?

* Let c = girt or circumference, and l = length of the tree.

Then $\frac{c}{4} \times \frac{c}{4} \times l = \frac{c^2 l}{16}$ = content of the tree according to the rule.

And $\frac{c^2}{4 \times 3.1416} \times l = \frac{c^2 l}{12.5664}$ = true content, according to the rule for finding the content of a cylinder.

But $\frac{c^2 l}{12.5664}$ differs from $\frac{c^2 l}{16}$ by nearly $\frac{1}{4}$ part of the whole, and therefore the rule is exceedingly erroneous.

When the tree is tapering, the mean girt is found in the same manner as in board measure. Or if the tree be very irregular, the best way is to divide it into a certain number of lengths, and find the content of each part separately.

When trees have their bark on, an allowance is generally made, by deducting so much from the girt as is judged sufficient to reduce it to such a circumference as it would have without its bark. In oak this allowance is about $\frac{1}{16}$ or $\frac{1}{12}$ part of the girt; but for elm, beech, ash, &c. whose bark is not so thick, the deduction ought to be less.

TIMBER MEASURE.

By Decimals.

$3.25 = 39$ inches.

3.25

1625

650

975

10.5625

$$9.75 = 9\frac{3}{4} \text{ feet.}$$

528125

739375

950625

$$102.984375 = \text{solidity.}$$

By Cross Multiplication.

f. in.

3 3 = 39 inches.

3 3

9 9

99

to 6 9

$$9 \cdot 9 = 9\frac{1}{2} \text{ feet}$$

95 0 9

7 11 0 9

102 11 9 9 = *solidity*.

By the SLIDING RULE.

As 91 upon C : 12 upon D :: 39 upon D : 103
upon C, the content.

2. The

2. The length of a tree is 25 feet, and the girt throughout $2\frac{1}{2}$ feet: what is its solidity?

Ans. 9 feet 9 inches.

3. The length of a tree is $14\frac{1}{2}$ feet, and its girt in the middle 3.15 feet: required the solidity?

Ans. 9 feet nearly.

4. The girts of a tree in 4 different places is as follows: in the first place 5 feet 9 inches, in the second 4 feet 5 inches, in the third 4 feet 9 inches, and in the fourth 3 feet 9 inches; and the length of the whole tree is 15 feet: what is the solidity?

Ans. 20 feet 5 inches.

5. An oak tree is 45 feet 7 inches long, and its quarter girt 3 feet 8 inches; what is the solid content, allowing $\frac{1}{4}$ for the bark?

Ans. 515 feet, nearly.

RULE II.

Multiply the square of $\frac{1}{4}$ of the girt by twice the length, and the product will be the solidity, *extremely near the truth.*

By

* Let c = circumference, and l = length, as before.

Then $\frac{c}{5} \times \frac{c}{5} \times 2l = \frac{2c^2l}{25} = \frac{c^2l}{12.5}$ = content of the tree according to the rule.

And the true content is $= \frac{c^2l}{12.5664}$.

But $\frac{c^2l}{12.5}$ differs from $\frac{c^2l}{12.5664}$ by only about $\frac{1}{190}$ part of the whole, and is therefore sufficiently near the truth for any practical purpose.

This rule is full as easy in practice as the false one, and therefore ought to be generally used, since the ease of the other method is the only argument which is alledged for employing it.

By the SLIDING RULE.

EXAM-

Ru/. Multiply the square of the circumference by the length, and take $\frac{1}{11}$ of the product; from this last number subtract $\frac{1}{11}$ of itself, and the remainder will be the answer.

For $\frac{c^2}{11.5664} = \frac{7}{88}$ of c^2 very nearly; $= \frac{7}{88} - \frac{1}{88} \times$
 $c^2 = \frac{c^2}{11} - \frac{1}{8}$ of $\frac{c^2}{11}$, which is the same as the rule;
 and differs from the truth only 1 foot in 2400.

The following problems, as well as many other things in this section, were taken from *Dr. Hutton's Mensuration*. They are given to shew the artifices that may be used in measuring timber according to the false method now practised, and the absolute necessity there is of abolishing it.

PROBLEM I.

To find where a tree must be cut, so that the two parts, measured separately, shall produce a greater solidity than that of the whole tree, or any other two parts of it.

RULE. Cut it through exactly in the middle, or at $\frac{1}{2}$ of the length, and the two parts will measure the most possible.

Demon. Put G = greatest girth, g = least, x = girth at the section, L = whole length, and z = length to be cut off the left end.

Then, by similar figures, $L : x :: G - g : x - g$, or $x = \frac{Gx - Gg}{x - g} + g$.

But the content of the whole is only 1/8 (see). And the

And if the fluxion of this expression be put equal to nothing, and the value of x be substituted instead of it, there will result $x = \frac{1}{2}L$. Q. E. D.

Example.

EXAMPLES.

1. A piece of timber is $9\frac{1}{2}$ feet long, and $\frac{1}{3}$ of the girt is 2.6 feet: what is the solidity?

By

Example. Supposing a tree to girt 14 feet at the greater end, 2 feet at the less, and 8 feet in the middle, and that the length is 32 feet.

Then, by the common method, the whole tree measures only 128 feet.

But, when cut through the middle, the greater part measures 121 feet, and the less part 25 feet.

And the sum of these two parts is 146 feet, which exceeds the whole by 18 feet, and is the most that it can be made to measure by cutting it into two parts.

PROBLEM II.

To find where a tree must be cut, so that the part next the greater end may measure the most possible.

Rule. Cut it where the girt is $\frac{1}{3}$ of the greatest girt, and the greater end will measure to the most possible.

Demon. By using the same notation as in the last problem, we shall have $x = \frac{Gx - gx}{L} + g$, and $(G+x)^2 \times L - x = a$ maximum.

And, if this be put into fluxions, it will give $x = \frac{G - 3g}{G - g}$ $\times \frac{1}{3}L$, and $x = \frac{G - g}{L} \times x + g = \frac{1}{3}G$. Q. E. D.

Example. Taking here the same example as before, we shall have $12 : 8 :: \frac{32}{3} : 7\frac{1}{3} =$ length to be cut off, $24\frac{2}{3} =$ length of the remaining part, and $4\frac{2}{3} =$ girt at the section.

But the content of the whole tree is only 128 feet, and the content of the greater part $135\frac{1}{3}$ feet, which exceeds 128 by $7\frac{1}{3}$, and is the greatest possible.

N 4

Note.

By Decimals.

$$\begin{array}{r}
 2.6 \\
 2.6 \\
 \hline
 156 \\
 52 \\
 \hline
 6.76 \\
 9.75 \\
 \hline
 3380 \\
 4732 \\
 6084 \\
 \hline
 65.9100 \\
 2
 \end{array}$$

131.8200 = content.

By

Note. If the greatest girt do not exceed 3 times the least, the tree cannot be cut as is required by the problem; for when the least girt is exactly equal to $\frac{1}{3}$ of the greater, the tree already measures to the most possible.

P R O B L E M I I I.

To cut a tree, so that the part next the greater end may measure exactly the same as the whole tree.

Rule 1. Call the sum of the girts of the two ends s , and their difference d .

2. Multiply d by the sum of d and s , and from the square root of the product take the difference between d and $2s$.

3. Then as $2d$ is to this remainder, so is the whole length of the tree, to the length to be cut off the smaller end.

Demon.

By the SLIDING RULE.

As 19.15 upon C : 12 upon D :: $31\frac{1}{2}$ in. upon D : 332, the content upon C.

2. If the length of a tree be 24 feet, and the girt throughout 8 feet: what is the content?

Ans. 123 feet, nearly.

3. If a tree girt 14 feet at the thicker end, and 2 feet at the smaller end: required the solidity when the length is 24 feet?

Ans. $11\frac{1}{2}$ feet, nearly.

4. A tree girts in five different places as follows: in the first place 9.43 feet, in the second 7.92 feet, in the third 6.15 feet, in the fourth 4.74 feet, and in the fifth 3.16 feet; and the whole length is $17\frac{1}{2}$ feet: what is the solidity?

Ans. 54.4065 feet.

Demon. Using still the same notation, we shall have $L = L - x \times G + x^2$; and by substituting for x its value $\frac{dx}{L} + g$,

it will be $x = \frac{L}{2d} \times \sqrt{4d^2 + d \times d - 2d + d}$, and $x = \frac{\sqrt{4d^2 + d \times d - 2d + d}}{2}$. Q. E. D.

The rule may be illustrated by taking the same example as in the last problems.

9. The distance of the centers of two circles whose diameters are each 70, is equal to 50: what is the area of the space included by their circumferences? *Ans.* 1000.

10. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100: what is the radius of the semicircle?

1. **W**HAT difference is there between a floor 48 feet long, and 30 feet broad, and two others each of half the dimensions? *Ans.* 720 feet.

2. From a mahogany plank 26 inches broad, a yard and a half is to be sawed off; what distance from the end must the line be struck? *Ans.* 6.23 feet.

3. A joist is $8\frac{1}{2}$ inches deep, and $3\frac{1}{2}$ broad: what will be the dimensions of a scantling just as big again as the joist, that is $4\frac{1}{2}$ inches broad? *Ans.* 12.52 inches deep.

4. A roof is 34 feet 8 inches by 14 feet 6 inches, and is to be covered with lead at 8 lbs. to the foot: what will it come to at 18 s. per cwt? *Ans.* 23l. 19s. 10½d.

5. What is the side of that equilateral triangle, whose area cost as much paving at 8d. per foot, as the pallisading the three sides did at a guinea per yard? *Ans.* 72.746.

6. The two sides of an obtuse-angled triangle are 20 and 40 poles: what must the length of the third side be that the triangle may contain just an acre? *Ans.* 58.876, or 23.099.

7. If two sides of a triangle, whose area is 60√3, be 12 and 20: what is the third side? *Ans.* 28.

8. If an area of 24 be cut off from a triangle, whose three sides are 13, 14 and 15, by a line parallel to the longest side: what are the lengths of the sides including that area? *Ans.* $\frac{12}{5}\sqrt{14}$, $2\sqrt{14}$, and $\frac{13}{5}\sqrt{14}$.

9. The

9. The distance of the centers of two circles whose diameters are each 50, is equal to 30: what is the area of the space inclosed by their circumferences? *Ans.* 559.119.

10. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semi-circle, is equal to 100: what is the diameter of the semi-circle? *Ans.* 26.32148.

11. The four sides of a field, whose diagonals are equal to each other, are 25, 35, 31 and 19 poles, respectively: what is the area? *Ans.* 4.ac. 1.ro. 38 poles.

12. What is the length of a chord which cuts off $\frac{1}{4}$ of the area from a circle whose diameter is 289? *Ans.* 278.6716.

13. A cable which is 3 feet long, and 9 inches in compass, weighs 22 lbs. what will a fathom of that cable weigh whose diameter is 9 inches? *Ans.* 434.28 lbs.

14. A circular fish-pond is to be dug in a garden, that shall take up just half an acre: what must the length of the chord be that strikes the circle? *Ans.* 27.75 yards.

15. A carpenter is to put an oaken curb to a round well, at 8d. per foot square; the breadth of the curb is to be $7\frac{1}{2}$ inches, and the diameter within $13\frac{1}{2}$ feet: what will be the expence? *Ans.* 5s. 2 $\frac{1}{2}$ d.

16. Suppose the expence of paving a semi-circular plot, at 2s. 4d. per foot, amounted to 10l. what is the diameter of it? *Ans.* 14.7737.

17. Seven men bought a grinding-stone of 60 inches in diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter must each grind

* For an excellent geometrical construction of this question, see *Dr. Hutton's Diarian Miscellany*, page 54.

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down for his share? *Ans. The 1st. 4.4508, 2d. 4.8400, 3d. 5.3535, 4th. 6.0765, 5th. 7.2079, 7th. 9.3935, and the 8th. 22.6778.*

18. A gentleman has a garden 100 feet long, and 80 feet broad, and a gravel walk is to be made of an equal width half round it: what must the width of the walk be so as to take up just half the ground?

Ans. 25.968 feet.

19. In the midst of a meadow well stored with grass,

I took just an acre to tether my ass;

How long must the cord be, that feeding all round,

He may'nt graze less or more than an acre of ground?

Ans. 39.25073 yards.

20. A maltster has a kiln that is 16 feet 6 inches square; now he wants to pull it down, and build a new one that will dry three times as much at a time as the old one did: what must be the length of its side?

Ans. 28 feet 7 inches.

21. If a round cistern be 26.3 inches diameter, and 52.5 inches deep: how many inches diameter must a cistern be to hold twice the quantity, the depth being the same?

Ans. 37.19 inches.

22. A May-pole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole May-pole, supposing the length of the broken piece to be 39 feet?

Ans. 75 feet.

23. What will the diameter of a globe be, when the solidity and superficial content thereof are equal to each other?

Ans. 6.

24. How many three inch cubes may be cut out of a 12 inch cube?

Ans. 64.

25. A farmer borrowed part of a hay-rick of his neighbour, which measured 6 feet every way, and paid him back again by two equal cubical pieces, each

each of whose sides were three feet: Query, whether the lender was fully paid? *Ans.* He was paid $\frac{1}{2}$ part only.

26. What will the painting a conical church-spire come to at 8*d.* per yard; supposing the circumference of the base to be 64 feet, and the altitude 118 feet?

Ans. 14*l.* 0*s.* 8 $\frac{1}{2}$ *d.*

27. What will a marble frustum of a cone come to at 12*s.* per solid foot; the diameter of the greater end being 4 feet, that of the lesser end 1 $\frac{1}{2}$ feet, and the length of the slant side 8 feet?

Ans. 30*l.* 1*s.* 10 $\frac{1}{2}$ *d.*

28. The diameter of a legal Winchester bushel is 18 $\frac{1}{2}$ inches, and its depth 8 inches: what must the diameter of that bushel be whose depth is 7 $\frac{1}{2}$ inches?

Ans. 19.10671.

29. Three men bought a tapering piece of timber, which was the frustum of a square pyramid; one side of the greater end was 3 feet, one side of the lesser end 1 foot, and the length 18 feet: what is the length of each man's piece, supposing they paid equally, and are to have equal shares? *Ans.* 1*ft.* 3.269, 2*d.* 4.559, and the 3*d.* 10.172, reckoning from the greater end to the less.

30. Suppose the ball at the top of St. Paul's Church is 6 feet in diameter: what did the gilding of it come to at 3 $\frac{1}{2}$ *d.* per square inch?

Ans. 237*l.* 19*s.* 10 $\frac{1}{2}$ *d.*

31. A person wants a cylindric vessel of 3 feet deep, that shall hold twice as much as a vessel of 28 inches deep, and 46 inches in diameter: what must be the diameter of the vessel required?

Ans. 57.37 inches.

32. Two postors agreed to drink off a quart of strong beer between them, at two pulls, or a draught each; now, the first having given it a black eye, as it is called, or drunk till the surface of the liquor touched the opposite edge of the bottom, gave the remaining

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remaining part of it to the other: what was the difference of their shares, supposing the pot was the frustum of a cone, the depth being 5.7 inches. The diameter at the top 3.7 inches, and that of the bottom 4.23 inches? *Ans.* 7.67 cubic inches.

33. Three persons having bought a sugar-loaf, want to divide it equally amongst them by sections parallel to the base; it is required to find the altitude of each person's share, supposing the loaf to be a cone, whose height is 20 inches? *Ans.* 13.867 the upper part, 3.604 the middle part, and 2.528 the lower part.

34. How high above the surface of the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface? *Ans.* To the height of the earth's diameter.

35. A cubical foot of brass is to be drawn into a wire of $\frac{3}{16}$ of an inch in diameter; what will be the length of the wire, allowing no loss in the metal? *Ans.* 3520252.69329 inches.

35. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raise one foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet? *Ans.* $7\frac{2}{3}$ feet.

36. Of what diameter must the bore of a cannon be, which is cast for a ball of 24 lbs. weight, so that the diameter of the bore may be $\frac{1}{16}$ of an inch more than that of the ball? *Ans.* 5.757 inches.

37. At what height from the bottom must an upright cone be cut, so that the greatest cylinder possible may be formed from the lower part of it? *Ans.* At $\frac{1}{3}$ of the height.

38. One ev'ning I chanc'd with a Tinker to sit,
Whose tongue ran a great deal too fast for his wit;
He talk'd of his art with abundance of mettle;
So I ask'd him to make me a flat-bottom'd kettle:

Let

Let the top and the bottom diameters be,
In just such proportion as five is to three:
Twelve inches the depth I propos'd, and no more;
And to hold in ale gallons seven less than a score.
He promis'd to do't, and forthwith to work went;
But when he had done it he found it too scant.
He alter'd then, but too big he had made it;
For though it held right the diameters fail'd it:
Thus making it often too big, and too little;
The Tinker at last had quite spoil'd his kettle:
But he swears that he'll bring his said promise to
pass,

Or else that he'll spoil every ounce of his brass.
Now, to keep him from ruin, I pray find him out
The diameter's length, for he'll ne'er do it I doubt.

Ans. The top diameter is 14.64017, and the
bottom diameter 24.40028.

A TABLE

T A B L E

O F T H E

AREAS of the SEGMENTS of a CIRCLE,
 Whose Diameter is Unity, and supposed to be di-
 vided into 1000 equal Parts.

Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area
.001	.000042	.024	.004921	.047	.013392
.002	.000119	.025	.005230	.048	.013818
.003	.000219	.026	.005546	.049	.014247
.004	.000337	.027	.005867	.050	.014681
.005	.000470	.028	.006194	.051	.015119
.006	.000618	.029	.006527	.052	.015561
.007	.000779	.030	.006865	.053	.016007
.008	.000951	.031	.007209	.054	.016457
.009	.001135	.032	.007558	.055	.016911
.010	.001329	.033	.007913	.056	.017369
.011	.001533	.034	.008273	.057	.017831
.012	.001746	.035	.008638	.058	.018296
.013	.001968	.036	.009008	.059	.018766
.014	.002199	.037	.009383	.060	.019239
.015	.002438	.038	.009763	.061	.019716
.016	.002685	.039	.010148	.062	.020196
.017	.002940	.040	.010537	.063	.020680
.018	.003202	.041	.010931	.064	.021168
.019	.003471	.042	.011330	.065	.021659
.020	.003748	.043	.011734	.066	.022154
.021	.004031	.044	.012142	.067	.022652
.022	.004322	.045	.012554	.068	.023154
.023	.004618	.046	.012971	.069	.023659

The Areas of the Segments of a Circle.

Ver- sed Sine	Seg. Area	Ver- sed Sine	Seg. Area	Ver- sed Sine	Seg. Area
.070	.024168	.103	.042687	.136	.064074
.071	.024680	.104	.043296	.137	.064760
.072	.025195	.105	.043908	.138	.065449
.073	.025714	.106	.044522	.139	.066140
.074	.026236	.107	.045139	.140	.066833
.075	.026761	.108	.045759	.141	.067528
.076	.027289	.109	.046381	.142	.068225
.077	.027821	.110	.047005	.143	.068924
.078	.028356	.111	.047632	.144	.069625
.079	.028894	.112	.048262	.145	.070328
.080	.029435	.113	.048894	.146	.071033
.081	.029979	.114	.049528	.147	.071741
.082	.030526	.115	.050165	.148	.072450
.083	.031076	.116	.050804	.149	.073161
.084	.031629	.117	.051446	.150	.073874
.085	.032186	.118	.052090	.151	.074589
.086	.032745	.119	.052736	.152	.075306
.087	.033307	.120	.053385	.153	.076026
.088	.033872	.121	.054036	.154	.076747
.089	.034441	.122	.054689	.155	.077469
.090	.035011	.123	.055345	.156	.078194
.091	.035585	.124	.056003	.157	.078921
.092	.036162	.125	.056663	.158	.079649
.093	.036741	.126	.057326	.159	.080380
.094	.037323	.127	.057991	.160	.081112
.095	.037909	.128	.058658	.161	.081846
.096	.038496	.129	.059327	.162	.082582
.097	.039087	.130	.059999	.163	.083320
.098	.039680	.131	.060672	.164	.084059
.099	.040276	.132	.061348	.165	.084801
.100	.040875	.133	.062026	.166	.085544
.101	.041476	.134	.062707	.167	.086289
.102	.042080	.135	.063389	.168	.087036

The

The Areas of the Segments of a Circle;

Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area
.169	.087785	.202	.113426	.235	.140688
.170	.088535	.203	.114230	.236	.141537
.171	.089287	.204	.115035	.237	.142387
.172	.090041	.205	.115842	.238	.143238
.173	.090797	.206	.116650	.239	.144091
.174	.091554	.207	.117460	.240	.144944
.175	.092313	.208	.118271	.241	.145799
.176	.093074	.209	.119083	.242	.146655
.177	.093836	.210	.119897	.243	.147512
.178	.094601	.211	.120712	.244	.148371
.179	.095366	.212	.121529	.245	.149230
.180	.096134	.213	.122347	.246	.150091
.181	.096903	.214	.123167	.247	.150953
.182	.097674	.215	.123988	.248	.151816
.183	.098447	.216	.124810	.249	.152680
.184	.099221	.217	.125634	.250	.153546
.185	.099997	.218	.126459	.251	.154412
.186	.100774	.219	.127285	.252	.155280
.187	.101553	.220	.128113	.253	.156149
.188	.102334	.221	.128942	.254	.157019
.189	.103116	.222	.129773	.255	.157890
.190	.103900	.223	.130605	.256	.158762
.191	.104685	.224	.131438	.257	.159636
.192	.105472	.225	.132272	.258	.160510
.193	.106261	.226	.133108	.259	.161386
.194	.107051	.227	.133945	.260	.162263
.195	.107842	.228	.134784	.261	.163140
.196	.108636	.229	.135624	.262	.164019
.197	.109430	.230	.136465	.263	.164899
.198	.110226	.231	.137307	.264	.165780
.199	.111024	.232	.138150	.265	.166663
.200	.111823	.233	.138995	.266	.167546
.201	.112624	.234	.139841	.267	.168430

The Areas of the Segments of a Circle.

Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area
.268	.169315	.301	.199085	.334	.229801
.269	.170202	.302	.200003	.335	.230745
.270	.171089	.303	.200922	.336	.231689
.271	.171978	.304	.201841	.337	.232634
.272	.172867	.305	.202761	.338	.233580
.273	.173758	.306	.203683	.339	.234526
.274	.174649	.307	.204605	.340	.235473
.275	.175542	.308	.205527	.341	.236421
.276	.176435	.309	.206451	.342	.237369
.277	.177330	.310	.207376	.343	.238318
.278	.178225	.311	.208301	.344	.239268
.279	.179122	.312	.209227	.345	.240218
.280	.180019	.313	.210154	.346	.241169
.281	.180918	.314	.211082	.347	.242121
.282	.181817	.315	.212011	.348	.243074
.283	.182718	.316	.212940	.349	.244026
.284	.183619	.317	.213871	.350	.244980
.285	.184521	.318	.214802	.351	.245934
.286	.185425	.319	.215733	.352	.246889
.287	.186329	.320	.216666	.353	.247845
.288	.187234	.321	.217599	.354	.248801
.289	.188140	.322	.218533	.355	.249757
.290	.189047	.323	.219468	.356	.250715
.291	.189955	.324	.220404	.357	.251673
.292	.190864	.325	.221340	.358	.252631
.293	.191775	.326	.222277	.359	.253590
.294	.192684	.327	.223215	.360	.254550
.295	.193596	.328	.224154	.361	.255510
.296	.194509	.329	.225093	.362	.256471
.297	.195422	.330	.226033	.363	.257433
.298	.196337	.331	.226974	.364	.258395
.299	.197252	.332	.227915	.365	.259357
.300	.198168	.333	.228858	.366	.260320

The Areas of the Segments of a Circle.

Ver- sed Sine	Seg. Area	Ver- sed Sine	Seg. Area	Ver- sed Sine	Seg. Area
.367	.261284	.400	.293369	.433	.325900
.368	.262248	.401	.294349	.434	.326892
.369	.263213	.402	.295330	.435	.327882
.370	.264178	.403	.296311	.436	.328874
.371	.265144	.404	.297292	.437	.329866
.372	.266111	.405	.298273	.438	.330858
.373	.267078	.406	.299255	.439	.331850
.374	.268045	.407	.300238	.440	.332843
.375	.269013	.408	.301220	.441	.333836
.376	.269982	.409	.302203	.442	.334829
.377	.270951	.410	.303187	.443	.335822
.378	.271920	.411	.304171	.444	.336816
.379	.272890	.412	.305155	.445	.337810
.380	.273861	.413	.306140	.446	.338804
.381	.274832	.414	.307125	.447	.339798
.382	.275803	.415	.308110	.448	.340793
.383	.276775	.416	.309095	.449	.341787
.384	.277748	.417	.310081	.450	.342782
.385	.278721	.418	.311068	.451	.343777
.386	.279694	.419	.312054	.452	.344772
.387	.280668	.420	.313041	.453	.345768
.388	.281642	.421	.314029	.454	.346764
.389	.282617	.422	.315016	.455	.347759
.390	.283592	.423	.316004	.456	.348755
.391	.284568	.424	.316992	.457	.349752
.392	.285544	.425	.317981	.458	.350748
.393	.286521	.426	.318970	.459	.351745
.394	.287498	.427	.319959	.460	.352742
.395	.288476	.428	.320948	.461	.353739
.396	.289453	.429	.321938	.462	.354736
.397	.290432	.430	.322928	.463	.355732
.398	.291411	.431	.323918	.464	.356730
.399	.292390	.432	.324909	.465	.357727

The

The Areas of the Segments of a Circle.

Ver- fed Sine	Seg. Area	Ver- fed Sine	Seg. Area
.466	.358725	.484	.376702
.467	.359723	.485	.377701
.468	.360721	.486	.378701
.469	.361719	.487	.379700
.470	.362717	.488	.380700
.471	.363715	.489	.381699
.472	.364713	.490	.382699
.473	.365712	.491	.383699
.474	.366710	.492	.384699
.475	.367709	.493	.385699
.476	.368708	.494	.386699
.477	.369707	.495	.387699
.478	.370706	.496	.388699
.479	.371705	.497	.389699
.480	.372704	.498	.390699
.481	.373703	.499	.391699
.482	.374702	.500	.392699
.483	.375702		

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In pursuance of this Plan, of writing a book for the use of Schools, he has been very careful to make all his definitions and rules as concise as possible, consistent with that simplicity and clearness which is absolutely necessary in things of this nature; and afterwards to exemplify those rules with a sufficient number of examples; in selecting of which, he has made choice of such as are most likely to occur in business; and has also shewn, with great clearness and perspicuity, the reason of each rule in notes; and, in some instances, has illustrated and explained the examples, when he had reason to apprehend any difficulties would be found; or where any disputes have arisen between former authors; and, in this part of his work, Mr. Bonnycastle has shewn great ingenuity and judgment.

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It is a singular circumstance, that a book which is so generally recommended, should be so little known. The author, however, is not a stranger to the public; he has published several other works, which are well known, and which have been highly praised. The present work is a treatise on the science of numbers, and is written in a clear and concise manner. It is a book which every student of mathematics should have. The author has given us a scientific, as well as practical treatise of this useful branch of Learning; so that students of every class may have their desires thoroughly gratified. The Book and its Author, of whom we know nothing, but from this performance, we recommend to the protection of the Public.

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